



BPS and Semi-BPS Kink Families in N-Component Scalar Field Theories with Fourth-Degree Polynomial Potentials

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ABSTRACT

We develop a constructive theory of kink solutions in N-component scalar field theories in (1+1) dimensions whose self-interactions are at most quartic and which carry a discrete symmetry. Working inside the Bogomolny first-order framework, we build the potentials from superpotentials and classify the models that support continuous families of BPS and semi-BPS kinks. The two-component MSTB and BNRT systems, together with the recently introduced single-singular-point models, reappear as the N=2 slice of a much larger landscape. The central result is a fully explicit one-parameter family of N-component quartic potentials for which we determine the 2N vacua, the first-order semi-BPS equations, the embedded one-component kinks, the exact energy sum rules, and the closed-form transverse Poschl-Teller spectra. The composite defects are interpreted through exact energy budgets. Because the generic profiles are not elementary, we show that a physics-informed neural network integrates the first-order equations directly and comment on multi-field solitons in the life sciences.

1. Introduction

Topological defects are localized, finite-energy solutions of nonlinear field theories that behave as extended particles and that cannot relax to the vacuum, because a conserved topological charge separates them from the trivial configuration [1–3]. In (1 + 1) dimensions these defects are the kinks, and even the single-field ϕ^4 and ϕ^6 theories already exhibit a strikingly intricate dynamics of scattering, resonant energy exchange and long-lived oscillations [4–6]. The mathematical and physical literature

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on these objects is now vast, spanning cosmology, condensed matter and high-energy physics, and several monographs collect the core methods and phenomenology [7–10]. Historically the subject grew out of the realization, going back to Skyrme [11], that classical field configurations carrying a conserved winding can model particle-like excitations, an idea whose two-dimensional incarnation is the kink.

Passing from one to several scalar components enlarges the space of static solutions dramatically. The vacuum manifold acquires a richer geometry, the orbits in field space become genuinely multi-dimensional, and entire continuous families of degenerate kinks can appear where a single field would admit only an isolated solution [4, 12, 13]. The construction of such multi-component configurations has therefore been a productive enterprise for several decades, and the two-field MSTB and BNRT systems have become the standard laboratories in which the phenomenon of kink families is studied [13–15]. A particularly transparent organizing principle is the Bogomolny rearrangement [16]: whenever the potential $U(\phi)$ equals one half of the squared gradient of a superpotential $W(\phi)$, the second-order static equations collapse to a first-order gradient flow whose solutions saturate a topological energy bound. In a single field such a W always exists and is essentially unique; with several components its existence is a real restriction, and a given potential may admit none, one, or several non-trivially related superpotentials, a feature tied to the supersymmetric structure of these models [12, 17–19].

Quartic self-interactions hold a distinguished place in this circle of ideas. In $(1 + 3)$ dimensions a polynomial potential of degree four contains the highest-order operators compatible with power-counting renormalizability, while in the Landau-Ginzburg description of critical phenomena the free energy is generically truncated at fourth order. Multi-field ϕ^4 -type theories are thus the minimal setting in which non-trivial vacuum manifolds, spontaneous symmetry breaking and topological sectors can be explored while analytic control is retained. For two components this programme was recently brought to a systematic conclusion by Alonso-Izquierdo, González León, González-Parra and Martín-Vaquero [20], who classified the quartic $\mathbb{Z}_2 \times \mathbb{Z}_2$ models that admit kink families, recovered the classical MSTB [14] and BNRT [13] systems, and uncovered a genuinely new family generated by an irrational superpotential possessing a single singular point. They explicitly recorded two natural directions for further work: relaxing the quartic restriction to higher polynomial degree, and increasing the number of scalar components. The present paper carries out the second of these programmes in full, and a companion line of work on multi-field kink collisions shows how rich the resulting dynamics can be already for two fields [21].

The gap we address is therefore concrete. While the two-component theory is now essentially complete, there has been no systematic construction of the N -component quartic models that support BPS and semi-BPS kink families, no closed-form determination of their vacua, energies and fluctuation spectra, and no sharp criterion telling which one-component kink is stable and when it becomes the seed of a continuous family. We close this gap. Imposing the $\mathbb{Z}_2^N$ symmetry and a normalization that reproduces the standard ϕ^4 model on the first axis, the constructive superpotential method yields a complete answer and, in particular, an explicit infinite class of new models. We obtain a self-contained N -component Bogomolny framework in which BPS and semi-BPS kinks, energy sum rules and the matrix fluctuation operator are formulated for arbitrary N ; the N -component BNRT family generated by a cubic superpotential, whose four-vacuum two-field core grows into a $2N$ -vacuum structure hosting genuinely multi-component composite kinks; and, as the principal contribution, a fully explicit one-parameter family of N -component quartic potentials generated by an irrational superpotential with one singular point, for which we compute the $2N$ vacua, the first-order equations, the embedded one-component kinks, the energy sum rules and the complete Pöschl-Teller spectra in closed form. Two clean consequences follow: the one-component kink $\mathcal{K}_a$ is stable precisely when $\lambda_a > 4\lambda_b$ for all $b \neq a$, and a second zero mode appears exactly when $\lambda_b > \lambda_a$, signalling the onset of

a kink family. We also pin down the transverse continuum threshold and the number of bound fluctuation modes in closed form. Throughout, every two-component formula of [20] is recovered verbatim as the $N = 2$ slice, which serves as a stringent internal check.

A second thread of the paper is computational. The generic family member cannot be written in elementary terms, since the orbit equation integrates to special functions and the subsequent reparameterization in the spatial variable is not elementary. We show that a physics-informed neural network (PINN) [22, 23] provides a mesh-free, derivative-exact integrator for the first-order semi-BPS equations, sidestepping the delicate treatment of the singular point that mesh methods require. Physics-informed learning has grown rapidly into a broad methodology for differential equations, with mature software and a substantial body of analysis [24–27]. On the embedded ϕ^4 kink our implementation recovers the analytic profile to a maximum pointwise error below 2×10^{-6} , well beyond the accuracy reported for a plain first-order Adam scheme, and the same loss applies unchanged to the irrational models. Finally, multi-field kinks are not merely formal: coupled scalar solitons underlie the Davydov model of vibrational energy transport in protein chains [28] and the Heimburg-Jackson account of nerve-pulse propagation in lipid membranes [29], both of which involve several coupled order parameters interacting through quartic-type free energies of exactly the structure analysed here. Figure 1 summarizes the constructive pipeline that ties these threads together. A $\mathbb{Z}_2^N$ -symmetric superpotential $W(\phi; \lambda)$ that reduces to the ϕ^4 structure on the first axis fixes a positive semi-definite quartic potential $U = \frac{1}{2} \sum_a (\partial_a W)^2$. Three structural routes, namely a cubic polynomial, an irrational superpotential with one singular point, and an irrational superpotential with two singular points, lead to first-order (semi-)BPS equations whose heteroclinic solutions assemble into kink families carrying two zero modes, characterized by exact energy sum rules and the matrix fluctuation operator (Figure 1).

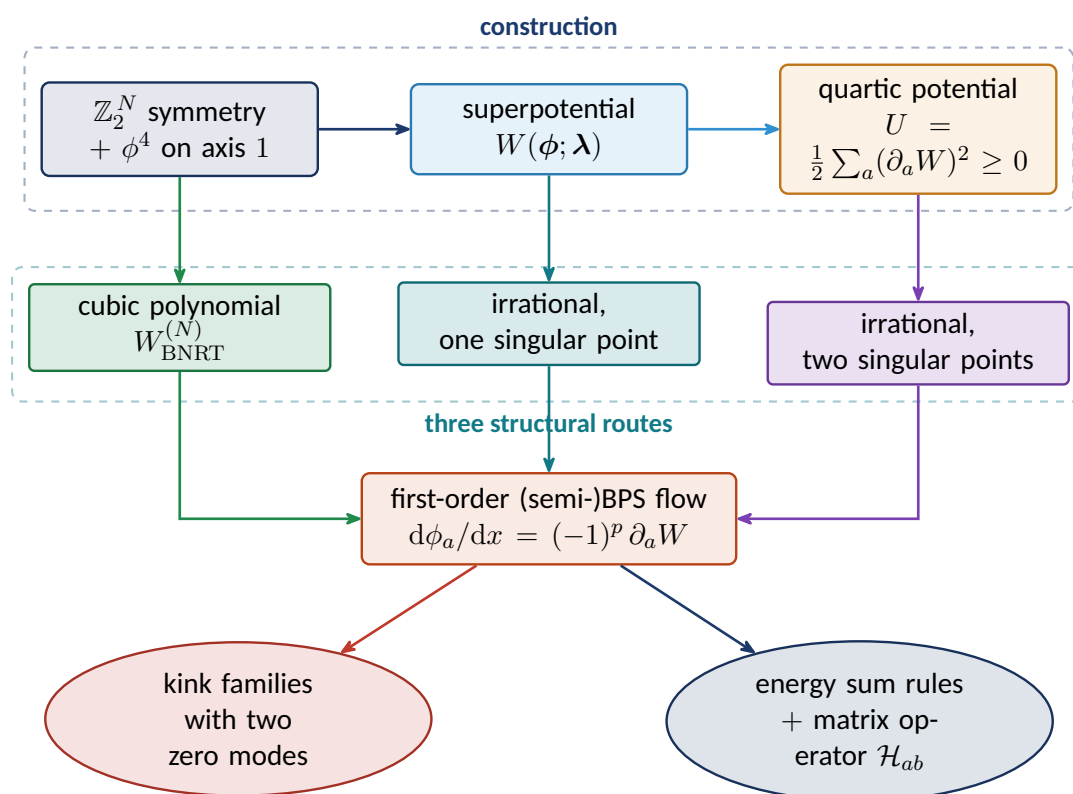


Fig. 1. The constructive Bogomolny pipeline used throughout

2. Preliminaries

We gather the objects and conventions used in the rest of the paper. Throughout, $\phi = (\phi_1, \dots, \phi_N)$ denotes a static map $\mathbb{R} \rightarrow \mathbb{R}^N$, a prime denotes d/dx , and repeated indices are not summed unless an explicit sum is shown. The definitions follow the standard treatments of classical solitons [2, 3, 7, 8] specialized to the $(1+1)$ -dimensional multi-field setting.

Definition 1. A $\mathbb{Z}_2^N$ -symmetric scalar model is the pair (ϕ, U) governed by the action (1) with a potential $U: \mathbb{R}^N \rightarrow [0, \infty)$ that is polynomial of degree at most four and invariant under each reflection $\phi_a \mapsto -\phi_a$. Its vacuum manifold is the zero set $\mathcal{M} = \{v \in \mathbb{R}^N : U(v) = 0\}$, assumed discrete.

Definition 2. A finite-energy static configuration is a map ϕ with $E[\phi] < \infty$, where E is the energy functional (3). Finiteness forces $\phi(\pm\infty) \in \mathcal{M}$, and the ordered pair $(\phi(-\infty), \phi(+\infty))$ labels the topological sector. A kink is a finite-energy static solution whose limits are two distinct elements of $\mathcal{M}$.

Definition 3. A superpotential for U is a function $W: \mathbb{R}^N \rightarrow \mathbb{R}$, differentiable off an isolated singular set, such that $U = \frac{1}{2} \sum_a (\partial_a W)^2$. The critical points of W off the singular set are vacua of U . When W is everywhere differentiable the model is called BPS; when the relevant orbit passes through a point at which W is singular, the model is called semi-BPS.

Definition 4. For a configuration with limits $\phi(\pm\infty)$ the Bogomolny bound is the number $T = |W(\phi(+\infty)) - W(\phi(-\infty))|$ in the BPS case, and the sum of the moduli of the superpotential increments across the singular point in the semi-BPS case. A configuration is BPS-saturating when $E[\phi] = T$, which holds exactly on solutions of the first-order system (6).

Definition 5. The second variation of E about a static solution ϕ is the matrix Schrödinger operator $\mathcal{H}_{ab}[\phi] = -\delta_{ab} d^2/dx^2 + \partial_{ab}^2 U(\phi)$ acting on $L^2(\mathbb{R}, \mathbb{R}^N)$. The solution is linearly stable when $\mathcal{H}[\phi] \geq 0$. A normalizable eigenfunction with eigenvalue $\omega^2 = 0$ is a zero mode; translational invariance always supplies the zero mode $\propto \phi'$, and any further zero mode signals a continuous moduli direction.

Definition 6. A Pöschl-Teller operator is a one-dimensional Schrödinger operator $-d^2/dx^2 + W_0 - \ell(\ell+1) \kappa^2 \operatorname{sech}^2(\kappa x)$. Its discrete spectrum is finite, the bound levels sit below the continuum threshold W_0 , and for integer ℓ the well is reflectionless. The transverse fluctuations of the kinks constructed below are all of this type.

With Definitions 1 to 6 in place we proceed to the general N -component Bogomolny structure, then to the two superpotential families, and finally to the spectral, energetic and data-driven analysis.

3. Bogomolny Structure of N -Component Theories

Consider N real scalar fields ϕ_a , $a = 1, \dots, N$, in a $(1+1)$ -Minkowskian spacetime with metric $\eta_{\mu\nu} = \operatorname{diag}\{1, -1\}$ and $x^0 \equiv t$, $x^1 \equiv x$. The dynamics follows from the action

$$S[\phi] = \int d^2x \left(\frac{1}{2} \sum_{a=1}^N \partial_\mu \phi_a \partial^\mu \phi_a - U(\phi) \right) \quad (1)$$

with $U(\phi) \geq 0$. The Euler-Lagrange equations are $\partial_t^2 \phi_a - \partial_x^2 \phi_a = -\partial U / \partial \phi_a$, and for static configurations they reduce to

$$\frac{d^2 \phi_a}{dx^2} = \frac{\partial U}{\partial \phi_a}, \quad a = 1, \dots, N \quad (2)$$

Reading x as a time, Eq. (2) is the Newtonian motion of a unit-mass particle in the inverted potential $-U$, and the static energy

$$E[\phi] = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \sum_{a=1}^N \left(\frac{d\phi_a}{dx} \right)^2 + U(\phi) \right] \quad (3)$$

is the mechanical action of that analogue problem. The configuration space splits into the topological sectors of Definition 2, labelled by ordered pairs of elements of the vacuum manifold $\mathcal{M}$.

The Bogomolny construction of Definition 3 applies whenever a superpotential W exists with

$$U(\phi) = \frac{1}{2} \sum_{a=1}^N \left(\frac{\partial W}{\partial \phi_a} \right)^2 \quad (4)$$

This guarantees $U \geq 0$ and identifies the vacua with the critical points of W . When W is differentiable the energy (3) rearranges into a sum of squares plus a boundary term.

Proposition 1. *Let W be differentiable along a finite-energy orbit with limits $\phi(\pm\infty) \in \mathcal{M}$. Then:*

$$E[\phi] = \frac{1}{2} \int_{-\infty}^{\infty} dx \sum_{a=1}^N \left(\frac{d\phi_a}{dx} - (-1)^p \frac{\partial W}{\partial \phi_a} \right)^2 + T, \quad T = |W(\phi(\infty)) - W(\phi(-\infty))| \quad (5)$$

for a suitable choice of $p \in \{0, 1\}$, so that $E \geq T$ with equality precisely on the solutions of the first-order BPS system

$$\frac{d\phi_a}{dx} = (-1)^p \frac{\partial W}{\partial \phi_a}, \quad a = 1, \dots, N \quad (6)$$

Proof. Complete the square in Eq. (3) using Eq. (4):

$\frac{1}{2} \sum_a (\phi'_a)^2 + U = \frac{1}{2} \sum_a (\phi'_a - (-1)^p \partial_a W)^2 + (-1)^p \sum_a \phi'_a \partial_a W$. The cross term is a total derivative $(-1)^p \sum_a \phi'_a \partial_a W = (-1)^p dW(\phi(x))/dx$ whose integral is $(-1)^p [W(\phi(\infty)) - W(\phi(-\infty))]$. Choosing p so that this boundary contribution is non-negative gives Eq. (5). The integral of a sum of squares is non-negative and vanishes if and only if each integrand vanishes, which is Eq. (6); on such solutions $E = T$. Solutions of Eq. (6) are first variations of the energy and therefore solve the second-order equations Eq. (2), as one checks by differentiating Eq. (6) and using Eq. (4). $\square$

The energy of a BPS kink depends only on its topological sector, $E = T = |W(\phi(\infty)) - W(\phi(-\infty))|$, and along the orbit the trajectory in the internal $\mathbb{R}^N$ obeys $d\phi_a/d\phi_b = (\partial_a W)/(\partial_b W)$. When W fails to be differentiable at an isolated point ϕ_0 through which the orbit passes, the rearrangement of Proposition 1 is applied on each side of ϕ_0 separately and the energy becomes

$$E[\phi] = |W(\phi(\infty)) - W(\phi_0)| + |W(\phi(-\infty)) - W(\phi_0)| \quad (7)$$

each branch obeying an independent sign in Eq. (6). These are the semi-BPS solutions of Definition 3.

To compare models on an equal footing we require that the restriction of U to the first axis reproduce the standard ϕ^4 model and that the theory be invariant under the N reflections $\phi_a \mapsto -\phi_a$,

$$U(\phi) \big|_{\phi_2=\dots=\phi_N=0} = \frac{1}{2}(1 - \phi_1^2)^2, \quad U(\dots, -\phi_a, \dots) = U(\dots, \phi_a, \dots) \quad (8)$$

Consequently every model possesses at least the two vacua $A_{\pm} = (\pm 1, 0, \dots, 0)$ and admits the embedded one-component ϕ^4 kink

$$\mathcal{K}_1(x) = ((-1)^p \tanh \bar{x}, 0, \dots, 0), \quad \bar{x} \equiv x - x_0 \quad (9)$$

There is no loss of generality in Eq. (8): a translation to the midpoint of two vacua followed by a rotation aligning the first axis reduces any generic quartic with at least two vacua to this form. We now propose two superpotential families, cubic polynomials that generate quartic interactions and lead to BPS kinks, and irrational superpotentials with singular points that lead to semi-BPS families.

4. Models with Cubic Polynomial Superpotentials

The most direct route to quartic interactions is a cubic superpotential. Imposing the structure (8) together with the $\mathbb{Z}_2^N$ symmetry fixes the admissible couplings and yields the N -component generalization of the BNRT system:

$$W_{\text{BNRT}}^{(N)}(\phi) = \frac{1}{3} \phi_1 \left(\phi_1^2 + 3 \sum_{a=2}^N \beta_a \phi_a^2 - 3 \right), \quad \beta_a \in \mathbb{R} \quad (10)$$

whose potential, computed from Eq. (4), is:

$$U_{\text{BNRT}}^{(N)}(\phi) = \frac{1}{2} \left(\phi_1^2 + \sum_{a=2}^N \beta_a \phi_a^2 - 1 \right)^2 + 2 \sum_{a=2}^N \beta_a^2 \phi_1^2 \phi_a^2 \quad (11)$$

For $N = 2$ this is precisely the BNRT model [13, 20]. The corresponding BPS equations read:

$$\frac{d\phi_1}{dx} = - \left(\phi_1^2 + \sum_{a \geq 2} \beta_a \phi_a^2 - 1 \right), \quad \frac{d\phi_a}{dx} = -2\beta_a \phi_1 \phi_a \quad (a \geq 2) \quad (12)$$

the sign choice $p = 1$ generating the $A_- \rightarrow A_+$ kink. For all $\beta_a > 0$ the model possesses the $2N$ vacua:

$$\mathcal{M} = \left\{ A_{\pm} = (\pm 1, \mathbf{0}), \quad B_{\pm}^{(a)} = \pm \beta_a^{-1/2} \mathbf{e}_a \quad (a = 2, \dots, N) \right\} \quad (13)$$

where $\mathbf{e}_a$ is the a -th coordinate unit vector. Dividing two transverse equations in (12) gives, along every orbit, $d\phi_a/d\phi_b = (\beta_a/\beta_b)(\phi_a/\phi_b)$, that is $\phi_a^{\beta_b} \propto \phi_b^{\beta_a}$. Hence whenever two transverse fields are simultaneously excited they remain locked by a power law, and the model supports genuinely multi-component composite kinks.

The geometry of these orbits is clearest in the two-field reduction. Restricting to the (ϕ_1, ϕ_2) plane with a single coupling $\beta \equiv \beta_2$, the BPS flow $\phi_1' = 1 - \phi_1^2 - \beta \phi_2^2$, $\phi_2' = -2\beta \phi_1 \phi_2$ is a planar gradient system with the four vacua $A_{\pm}$ and $B_{\pm}$. Figure 2 displays its phase portrait for $\beta = 0.30$: the streamlines of the field define a heteroclinic web, and the one-parameter pencil of orbits leaving A_- and arriving at A_+ is the AA kink family. Streamlines in Figure 2 are coloured by the field speed; the four vacua $A_{\pm}, B_{\pm}$ are marked, and the dark heteroclinic curves are members of the AA kink family leaving A_- and reaching A_+ at increasing transverse amplitude. As the impact parameter $p = \phi_2(0)$ at the symmetric centre grows from zero toward the value $\beta^{-1/2}$, the orbit bulges further into the transverse direction and its energy density splits from a single lump into two. Figure 3 shows this transition together with the orbits in field space; every member carries the same energy $E = \frac{4}{3}$, confirming the absence of a static inter-lump force. In Figure 3, the integrated energy equals $\frac{4}{3}$ for every member.

Returning to three components with $(\beta_2, \beta_3) = (0.30, 0.15)$, Figure 4 (left) shows a genuinely three-component kink obtained by integrating Eq. (12) outward from the symmetric centre $\phi_1(0) = 0$ with both transverse fields excited. The defect connects A_- to A_+ while ϕ_2 and ϕ_3 bulge in the core, its energy density carries the two-lump signature of a composite object, and its total energy is numerically $E = \frac{4}{3}$, equal to that of the embedded kink $\mathcal{K}_1$. The right panel collects a one-parameter pencil of such orbits in the internal space, each connecting $A_{\mp}$.

Remark 1. As in the two-field case [20], moving the active field in Eq. (10) to another axis produces equivalent models related by a permutation of axes and a rescaling. Negative couplings remove the corresponding transverse vacua and the kink variety collapses to $\mathcal{K}_1$, so only $\beta_a > 0$ is of interest here.

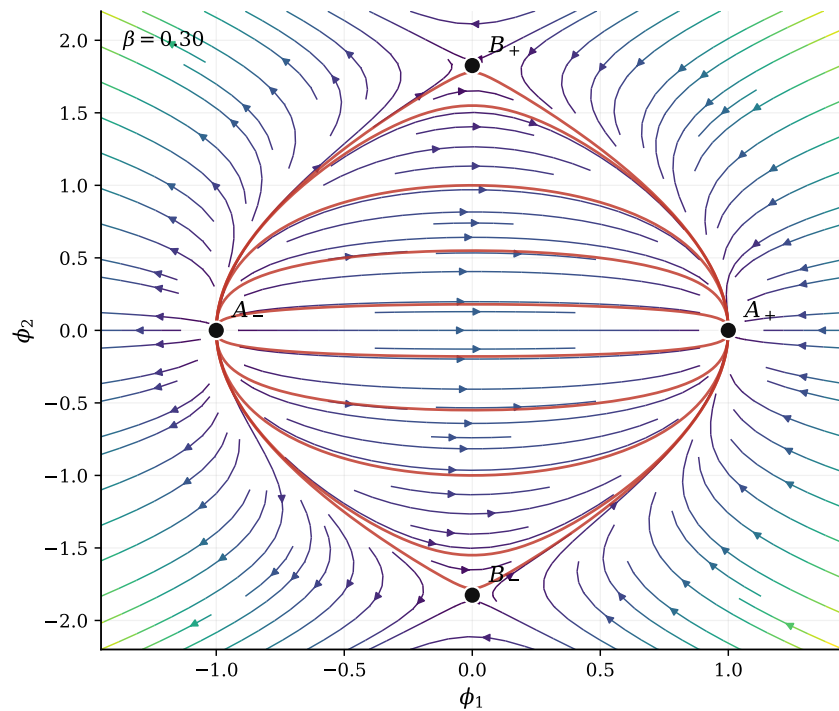


Fig. 2. Phase portrait of the BPS flow of the two-field BNRT reduction with $\beta = 0.30$

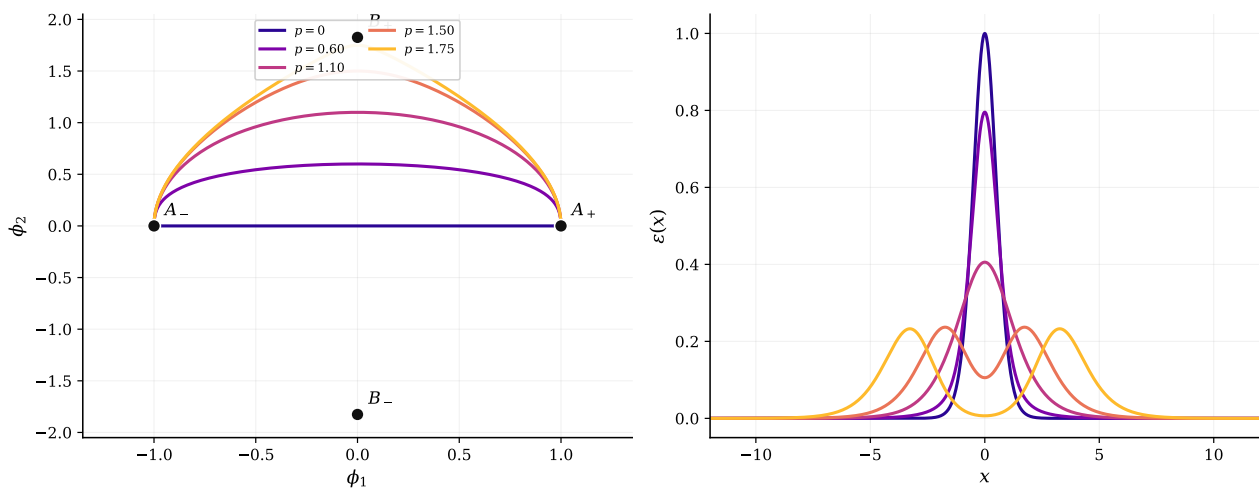


Fig. 3. The AA kink family of the two-field BNRT model, $\beta = 0.30$; left: orbits in the internal plane for impact parameters $p = \phi_2(0)$ ranging from the straight one-component path $p = 0$ toward the deformed orbits that approach the B vacua; right: the corresponding energy densities, which separate continuously from a single central lump into two symmetric lumps as p grows

5. A New Family: Irrational Superpotentials with One Singular Point

We now turn to the principal result. Following the single-singular-point construction of [20], we adopt for N components the irrational ansatz

$$W_N(\phi) = \frac{1}{3} \|\phi\| \left(\sum_{a=1}^N \lambda_a \phi_a^2 - 3 \right), \quad \|\phi\| = \sqrt{\sum_a \phi_a^2}, \quad \lambda_1 \equiv 1, \quad (14)$$

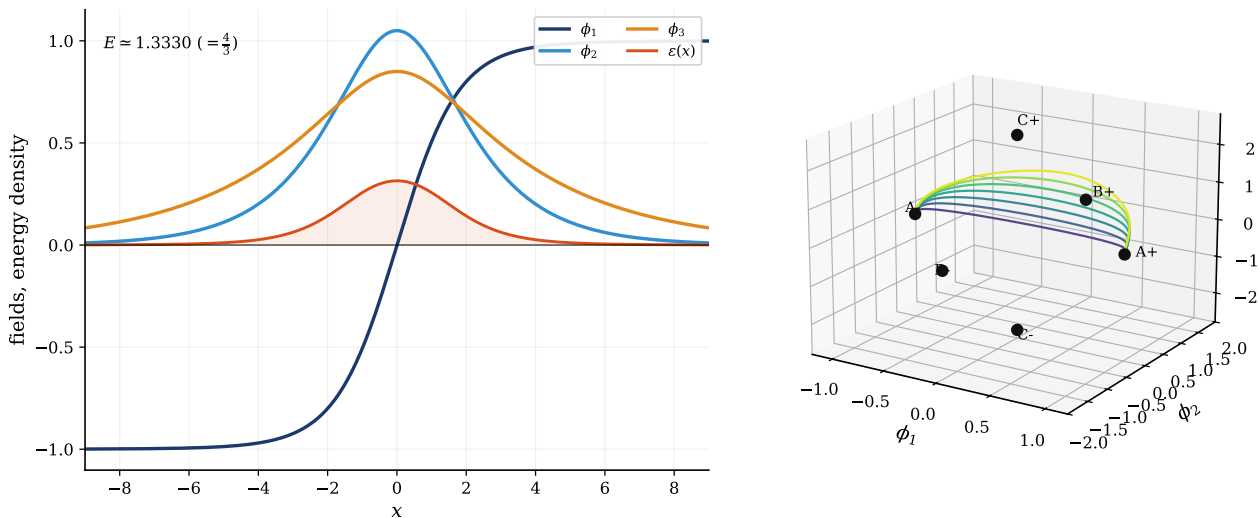


Fig. 4. The $N = 3$ BNRT model with $(\beta_2, \beta_3) = (0.30, 0.15)$; left: a genuinely three-component BPS kink in the AAA sector; the dark blue, light blue and amber curves are ϕ_1, ϕ_2, ϕ_3 , and the orange curve is the energy density $\varepsilon(x)$, whose two lumps reveal the composite nature of the solution; right: a family of BPS orbits in the internal space (ϕ_1, ϕ_2, ϕ_3) connecting the vacua $A_{\mp}$, with the six vacua $A_{\pm}, B_{\pm}, C_{\pm}$ marked

which is singular precisely at the origin $\phi = 0$. Substituting Eq. (14) into Eq. (4) and simplifying yields a remarkably compact closed form.

Proposition 2. The superpotential (14) generates the positive semi-definite quartic potential:

$$U_N(\phi) = \sum_{a=1}^N \frac{1}{2} (1 - \lambda_a \phi_a^2)^2 + \frac{1}{9} \sum_{1 \leq a < b \leq N} (2\lambda_a + \lambda_b)(\lambda_a + 2\lambda_b) \phi_a^2 \phi_b^2 - \frac{N-1}{2} \quad (15)$$

with $\lambda_1 = 1$. For $N = 2$ this reduces exactly to the two-field potential of [20], $U_2 = \frac{1}{2}(1 - \phi_1^2)^2 + \frac{1}{2}(1 - \lambda\phi_2^2)^2 + \frac{1}{9}(2 + \lambda)(1 + 2\lambda)\phi_1^2\phi_2^2 - \frac{1}{2}$ with $\lambda \equiv \lambda_2$.

Proof. Write $W_N = \frac{1}{3}r(Q - 3)$ with $r = \|\phi\|$ and $Q = \sum_a \lambda_a \phi_a^2$. Then $\partial_a W_N = \frac{1}{3}[(\phi_a/r)(Q - 3) + 2\lambda_a \phi_a r]$, and squaring and summing:

$$\sum_a (\partial_a W_N)^2 = \frac{1}{9}[(Q - 3)^2 + 4(Q - 3)Q + 4SR] = \frac{1}{9}[5Q^2 - 18Q + 9 + 4SR]$$

where $S = \sum_a \phi_a^2 = r^2$ and $R = \sum_a \lambda_a^2 \phi_a^2$, the three contributions coming respectively from $\sum_a \phi_a^2/r^2 = 1$, from $\sum_a \lambda_a \phi_a^2 = Q$, and from $\sum_a \lambda_a^2 \phi_a^2 r^2 = SR$. Hence $U_N = \frac{1}{18}[5Q^2 - 18Q + 9 + 4SR]$. Now $Q^2 = \sum_{a,b} \lambda_a \lambda_b \phi_a^2 \phi_b^2$ and $SR = \sum_{a,b} \lambda_b^2 \phi_a^2 \phi_b^2$, so the coefficient of ϕ_a^4 is $\frac{1}{18}(5\lambda_a^2 + 4\lambda_a^2) = \frac{1}{2}\lambda_a^2$, and for $a \neq b$ the coefficient of $\phi_a^2 \phi_b^2$ is $\frac{1}{18}(10\lambda_a \lambda_b + 4\lambda_a^2 + 4\lambda_b^2) = \frac{1}{9}(2\lambda_a + \lambda_b)(\lambda_a + 2\lambda_b)$. The linear term $\frac{1}{18}(-18Q) = -\sum_a \lambda_a \phi_a^2$ and the constant $\frac{1}{18} \cdot 9 = \frac{1}{2}$ combine with the quartic diagonal into $\sum_a \frac{1}{2}(1 - \lambda_a \phi_a^2)^2$ after subtracting the constant $\frac{N-1}{2}$ needed to balance the N unit contributions of the squares. This is (15); the identity has been confirmed symbolically for $N = 2, 3, 4$. $\square$

Two structural features deserve emphasis. First, the cross-coupling $(2\lambda_a + \lambda_b)(\lambda_a + 2\lambda_b)$ is the same symmetric kernel that governs the two-field interaction, now evaluated on every pair of axes; the N -component model is thus assembled from pairwise blocks together with the single-field ϕ^4 structure on each axis. Second, the construction retains the full $\mathbb{Z}_2^N$ symmetry.

The vacua of U_N are the critical points of W_N off the singular set. For $\lambda_a > 0$ they are the $2N$ axial points:

$$\mathcal{M} = \{ V_{\pm}^{(a)} = \pm \lambda_a^{-1/2} e_a : a = 1, \dots, N \}, \quad U_N(V_{\pm}^{(a)}) = 0 \quad (16)$$

with $V_{\pm}^{(1)} = A_{\pm} = (\pm 1, 0)$. We write $A, B, C, \dots$ for the vacua on the first, second, third axes, and so on. The origin is not a vacuum. Evaluating Eq. (15) at $\phi = 0$, each axial square contributes $\frac{1}{2}(1-0)^2 = \frac{1}{2}$, the cross-terms vanish, and the additive constant is $-\frac{N-1}{2}$, so that:

$$U_N(0) = \frac{N}{2} - \frac{N-1}{2} = \frac{1}{2} \neq 0 \quad \text{for every } N \quad (17)$$

independently of the couplings and of the number of components. The origin is the singular point of W_N through which the semi-BPS orbits pass; for $N = 2$ the value $\frac{1}{2}$ coincides numerically with $\frac{N-1}{2}$, but the two expressions differ for every $N \geq 3$, and it is Eq. (17) that holds in general. Figure 5 contrasts the four-vacuum square of $N = 2$ with the six-vacuum octahedron of $N = 3$ for $(\lambda_2, \lambda_3) = (\frac{1}{2}, 2)$.

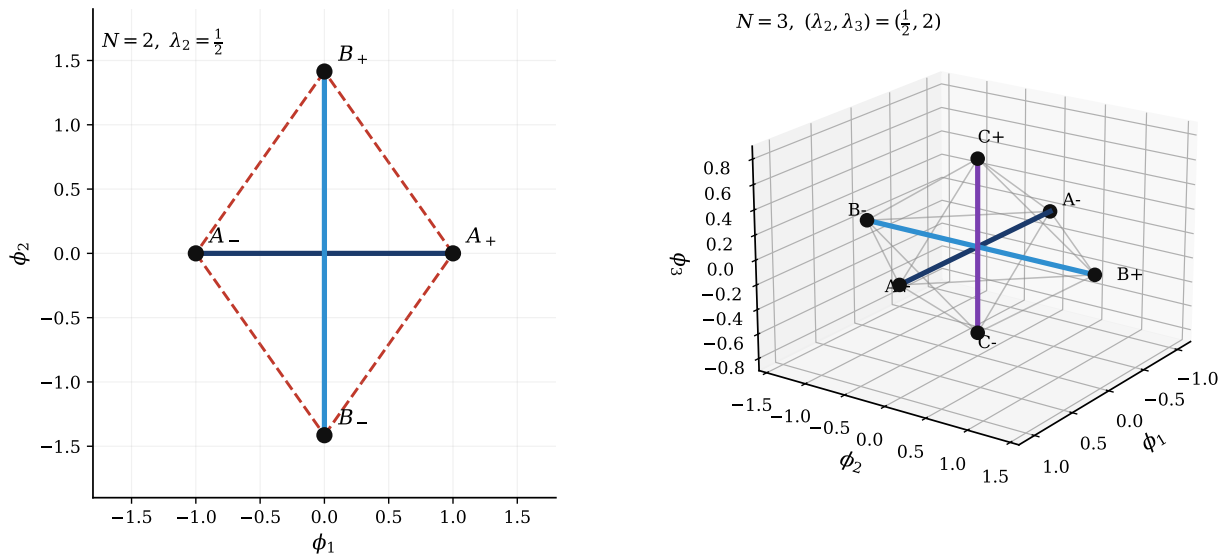


Fig. 5. Vacuum manifold of the irrational model (15); left: the four vacua of $N = 2$ for $\lambda_2 = \frac{1}{2}$; the dark and light blue segments are the one-component AA and BB kinks and the red edges the basic AB kinks; right: the six vacua of $N = 3$ for $(\lambda_2, \lambda_3) = (\frac{1}{2}, 2)$ form an octahedron; the coloured axes carry the one-component kinks $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and the faint edges the basic two-vacuum kinks

Figure 6 displays two orthogonal slices of U_3 . The slice $\phi_3 = 0$ (left) coincides with the two-field potential $U_2(\lambda_2)$, while $\phi_2 = 0$ (right) coincides with $U_2(\lambda_3)$. The differing well depths reflect the asymmetry $\lambda_2 \neq \lambda_3$. The degenerate minima sit at the vacua (16).

The semi-BPS equations derived from Eq. (14) are:

$$\frac{d\phi_a}{dx} = (-1)^p \frac{\phi_a}{3\|\phi\|} \left(\sum_b \lambda_b \phi_b^2 + 2\lambda_a \sum_b \phi_b^2 - 3 \right), \quad a = 1, \dots, N \quad (18)$$

the origin being a singularity of the flow. For $N = 2$ these are exactly the equations of [20]. Each axis carries an exact one-component kink that solves both Eq. (18) and the second-order equations (2):

$$\mathcal{K}_a(x) = \left(0, \dots, \underbrace{(-1)^p \lambda_a^{-1/2} \tanh(\sqrt{\lambda_a} x)}_{a\text{-th slot}}, \dots, 0 \right), \quad a = 1, \dots, N \quad (19)$$

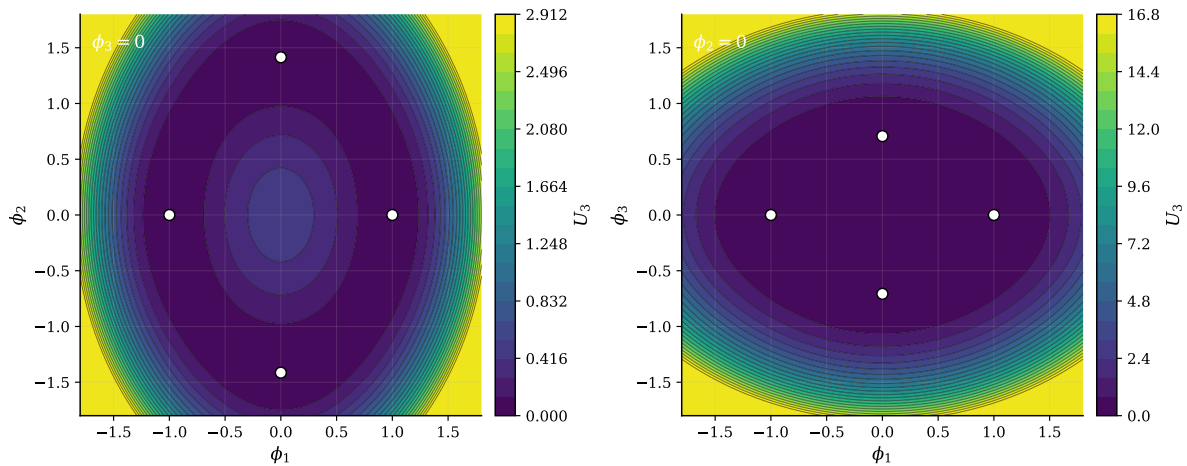


Fig. 6. Two orthogonal slices of the three-component potential U_3 for $(\lambda_2, \lambda_3) = (\frac{1}{2}, 2)$

with $\mathcal{K}_1$ the ϕ^4 kink Eq. (9). Figure 7 displays $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ together with their single-lump energy densities for $(\lambda_2, \lambda_3) = (\frac{1}{2}, 2)$. The amplitude and width are both $\lambda_a^{-1/2}$. The numerically integrated energies $\frac{4}{3}, \frac{4}{3\sqrt{\lambda_2}}, \frac{4}{3\sqrt{\lambda_3}}$ agree with Eq. (20) to machine precision.

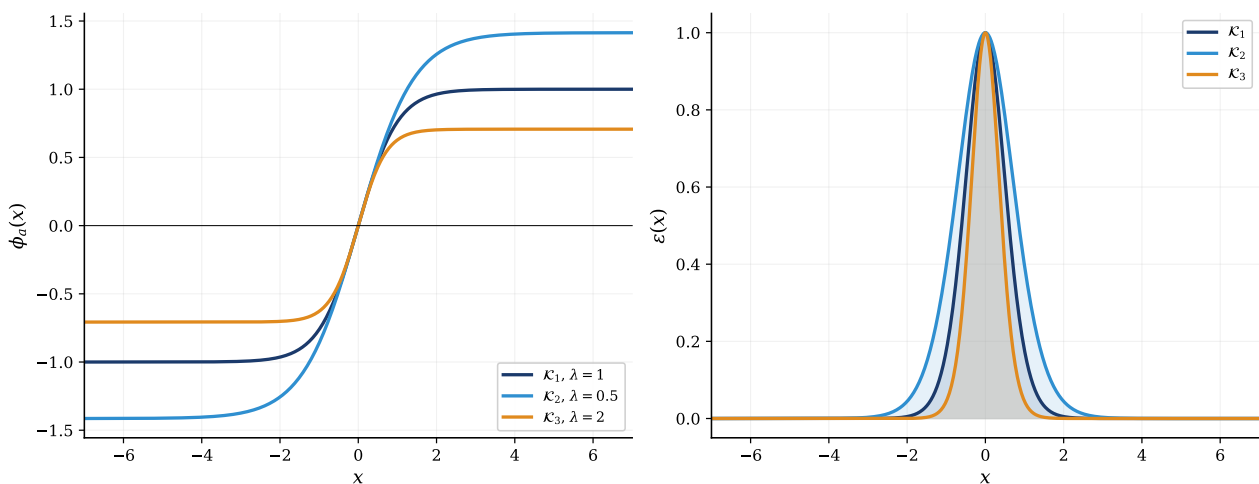


Fig. 7. The exact one-component kinks $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ of the irrational $N = 3$ model (left) and their energy densities $\varepsilon(x)$ (right), for $(\lambda_2, \lambda_3) = (\frac{1}{2}, 2)$

Evaluating the superpotential at the vacua gives $W_N(V_{\pm}^{(a)}) = -\frac{2}{3}\lambda_a^{-1/2}$ and $W_N(\mathbf{0}) = 0$. Using Proposition 1 together with Eq. (7), the one-component kinks pass through the singular origin and are semi-BPS, whereas the basic two-vacuum kinks $\mathcal{K}_{V^{(a)}V^{(b)}}$ connect adjacent axes directly and are BPS:

$$E[\mathcal{K}_a] = \frac{4}{3\sqrt{\lambda_a}}, \quad E[\mathcal{K}_{V^{(a)}V^{(b)}}] = \frac{2}{3} \left| \frac{1}{\sqrt{\lambda_a}} - \frac{1}{\sqrt{\lambda_b}} \right| \quad (20)$$

These combine into exact energy sum rules. Whenever $0 < \lambda_a < 1$ the one-component kink $\mathcal{K}_a$ sits at the centre of a continuous family $\mathcal{K}_{V^{(a)}V^{(a)}}(x; \kappa)$, which decomposes as:

$$E[\mathcal{K}_{V^{(a)}V^{(a)}}(x; \kappa)] = 2 E[\mathcal{K}_{V^{(1)}V^{(a)}}] + E[\mathcal{K}_1] = \frac{4}{3\sqrt{\lambda_a}} \quad (21)$$

that is, a composite of two basic $AV^{(a)}$ lumps placed symmetrically about a central $\mathcal{K}_1$ lump, the modulus κ fixing their separation. Figure 8 displays this budget. The stacked bars on the left show the exact decomposition (21) for several couplings, with the quadrature value of $E[\mathcal{K}_a]$ superposed. The curves on the right exhibit the additive relation across the whole range $0 < \lambda_a < 1$.

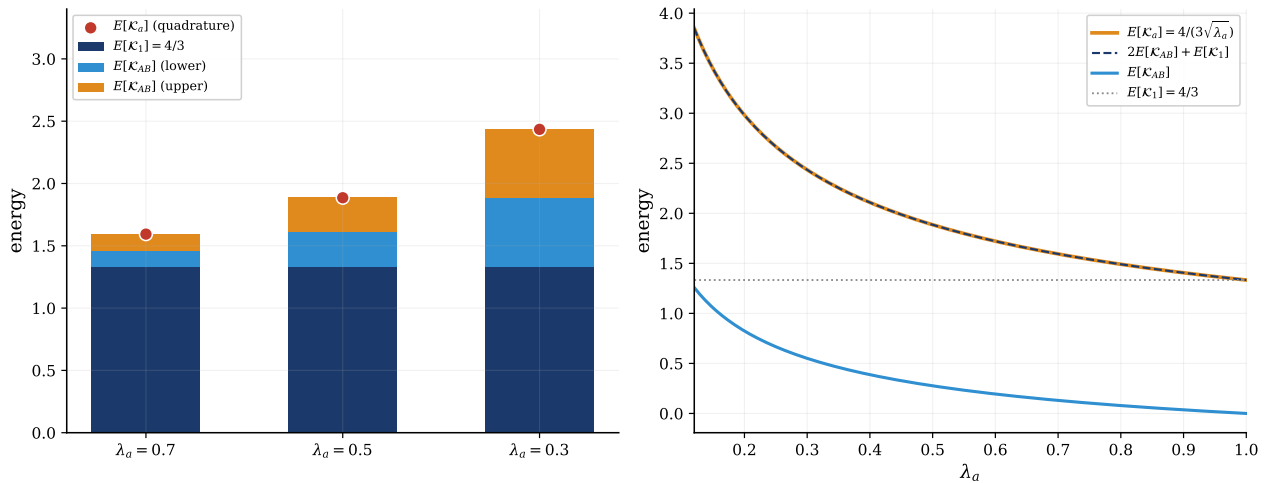


Fig. 8. Energy budget of the axial family in the irrational model; left: the exact sum rule (21) as stacked bars for $\lambda_a = 0.7, 0.5, 0.3$, the central $\mathcal{K}_1$ lump of energy $\frac{4}{3}$ topped by two $AV^{(a)}$ lumps; the red markers are the directly integrated energies $E[\mathcal{K}_a]$; right: the same quantities as functions of λ_a , showing that $2E[\mathcal{K}_{AV^{(a)}}] + E[\mathcal{K}_1]$ reproduces $E[\mathcal{K}_a] = 4/(3\sqrt{\lambda_a})$ identically

Table 1 collects the energies and lump content (Figure 8).

Table 1

Energy budget and composite (lump) interpretation in the irrational model for $0 < \lambda_a < 1$

Object	Sector	Energy	Lump content
$\mathcal{K}_1$	AA	$\frac{4}{3}$	one lump (fundamental)
$\mathcal{K}_{AV^{(a)}}$	$AV^{(a)}$	$\frac{2}{3}(\lambda_a^{-1/2} - 1)$	one lump (fundamental)
$\mathcal{K}_{V^{(a)}V^{(a)}}(x; \kappa)$	$V^{(a)}V^{(a)}$	$\frac{4}{3}\lambda_a^{-1/2}$	$2\mathcal{K}_{AV^{(a)}} + \mathcal{K}_1$ (three lumps)

Perturbing a static solution and linearizing Eq. (2) yields the matrix Schrödinger operator of Definition 5:

$$\mathcal{H}_{ab}[\phi] = -\delta_{ab} \frac{d^2}{dx^2} + \frac{\partial^2 U_N}{\partial \phi_a \partial \phi_b} \Big|_{\phi} \quad (22)$$

Evaluated on the one-component kink $\mathcal{K}_a$ the Hessian is diagonal: the longitudinal entry reproduces the ϕ^4 fluctuation operator, and each transverse direction $b \neq a$ contributes a Pöschl-Teller well in the sense of Definition 6. After rescaling $y = \sqrt{\lambda_a} x$ the transverse spectrum toward axis b is, in closed form,

$$\omega_n^2(\mathcal{K}_a; b) = -\frac{1}{3}(n-1)(3n\lambda_a + \lambda_a - 4\lambda_b), \quad n = 0, 1, \dots, n < \frac{\lambda_a + 2\lambda_b}{3\lambda_a}. \quad (23)$$

For $a = 1$ with $\lambda_1 = 1$ this is precisely the spectrum of [20], $\omega_n^2 = -\frac{1}{3}(n-1)(3n - 4\lambda_b + 1)$. For $a = b$ it reduces to the ϕ^4 shape spectrum. Eq. (23) has several immediate consequences, which we record as propositions.

Proposition 3. The lowest transverse eigenvalue of $\mathcal{K}_a$ toward axis b is $\omega_0^2 = \frac{1}{3}(\lambda_a - 4\lambda_b)$. Hence $\mathcal{K}_a$ is linearly stable if and only if:

$$\lambda_a > 4\lambda_b \quad \text{for every } b \neq a. \quad (24)$$

In particular $\mathcal{K}_1$ is stable if and only if $\lambda_b < \frac{1}{4}$ for all $b \geq 2$

Proof. Setting $n = 0$ in Eq. (23) gives $\omega_0^2 = -\frac{1}{3}(-1)(\lambda_a - 4\lambda_b) = \frac{1}{3}(\lambda_a - 4\lambda_b)$, the ground level of the Pöschl-Teller well. Since the longitudinal ϕ^4 operator is non-negative with a single zero mode, $\mathcal{H}[\mathcal{K}_a] \geq 0$ holds if and only if every transverse ground level is non-negative, which is Eq. (24). The values $\lambda_1 = 1$ give the stated corollary. $\square$

Proposition 4. The transverse continuum of $\mathcal{H}[\mathcal{K}_a]$ toward axis b begins at:

$$\omega_{\text{th}}^2(\mathcal{K}_a; b) = \frac{4(\lambda_a - \lambda_b)^2}{9\lambda_a} \quad (25)$$

which for $\mathcal{K}_1$ equals $4(1 - \lambda_b)^2/9$. The lowest mode lies strictly below threshold for all couplings, with gap $\omega_{\text{th}}^2 - \omega_0^2 = (2\lambda_b + \lambda_a)^2/(9\lambda_a)$ along the $\mathcal{K}_1$ channel reducing to $(2\lambda_b + 1)^2/9 > 0$, so the ground level is always a genuine bound state.

Proof. The threshold is the asymptotic value of the transverse potential, namely the transverse mass squared at the vacuum, $\omega_{\text{th}}^2 = \lim_{x \rightarrow \infty} \partial_{bb}^2 U_N(\mathcal{K}_a)$. Using Eq. (15) the second derivative toward axis b on the kink $\mathcal{K}_a$ equals $-2\lambda_b + \frac{2}{9\lambda_a}(2\lambda_a + \lambda_b)(\lambda_a + 2\lambda_b) \tanh^2(\sqrt{\lambda_a}x)$, whose $x \rightarrow \infty$ limit simplifies to $4(\lambda_a - \lambda_b)^2/(9\lambda_a)$. Subtracting $\omega_0^2 = (\lambda_a - 4\lambda_b)/3$ gives the stated gap, manifestly positive. $\square$

Proposition 5. The $n = 1$ transverse mode of $\mathcal{K}_a$ toward axis b is an exact zero mode, $\omega_1^2 = 0$, and it is normalizable precisely when $(\lambda_a + 2\lambda_b)/(3\lambda_a) > 1$, that is $\lambda_b > \lambda_a$. The appearance of this second zero mode, additional to the translational one, signals that $\mathcal{K}_a$ belongs to a continuous one-parameter family generated by an orbit deformation in the ϕ_b direction. Consequently the number of transverse bound states toward axis b equals the number of integers $n \geq 0$ with $n < (\lambda_a + 2\lambda_b)/(3\lambda_a)$.

Proof. Setting $n = 1$ in Eq. (23) gives $\omega_1^2 = 0$ identically. The Pöschl-Teller bound levels are indexed by n with $n < \ell_a = (\lambda_a + 2\lambda_b)/(3\lambda_a)$, so the $n = 1$ level is present in the discrete spectrum exactly when $\ell_a > 1$, equivalent to $\lambda_b > \lambda_a$. A normalizable zero mode is a tangent direction to a family of static solutions of equal energy, which is the asserted family. Counting the admissible integers n gives the number of bound states. $\square$

Figure 9 maps the spectral content of $\mathcal{K}_1$ as the transverse coupling varies. The blue curve is the lowest eigenvalue $\omega_0^2 = (1 - 4\lambda)/3$, which crosses zero at $\lambda = \frac{1}{4}$ and renders $\mathcal{K}_1$ unstable beyond it; the amber line is the $n = 1$ zero mode, bound only for $\lambda > 1$; the grey curve is the continuum threshold $4(1 - \lambda)^2/9$, above which the shaded region is continuum. Red markers are finite-difference eigenvalues, confirming Eq. (23) and Eq. (25) (Figure 9).

Figure 10 resolves the bound modes themselves. Figure 11 partitions the (λ_2, λ_3) plane into the regions where each one-component kink is the stable ground state. The shaded region indicates where $\mathcal{K}_1$ is the stable ground state, namely $\lambda_2, \lambda_3 < \frac{1}{4}$, bounded by the lines $\lambda_a = 4\lambda_b$ of Proposition 3. The corresponding wedges for $\mathcal{K}_2$ and $\mathcal{K}_3$ lie at large λ_2 or λ_3 . The representative point $(\frac{1}{2}, 2)$ used throughout is marked (Figure 11).

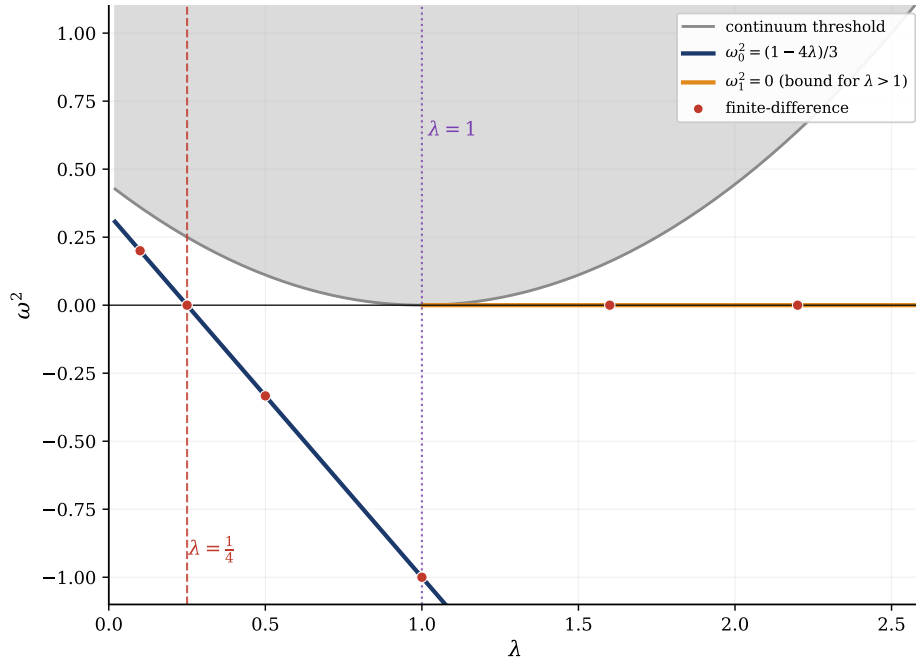


Fig. 9. Transverse fluctuation spectrum of $\mathcal{K}_1$ as a function of the coupling λ

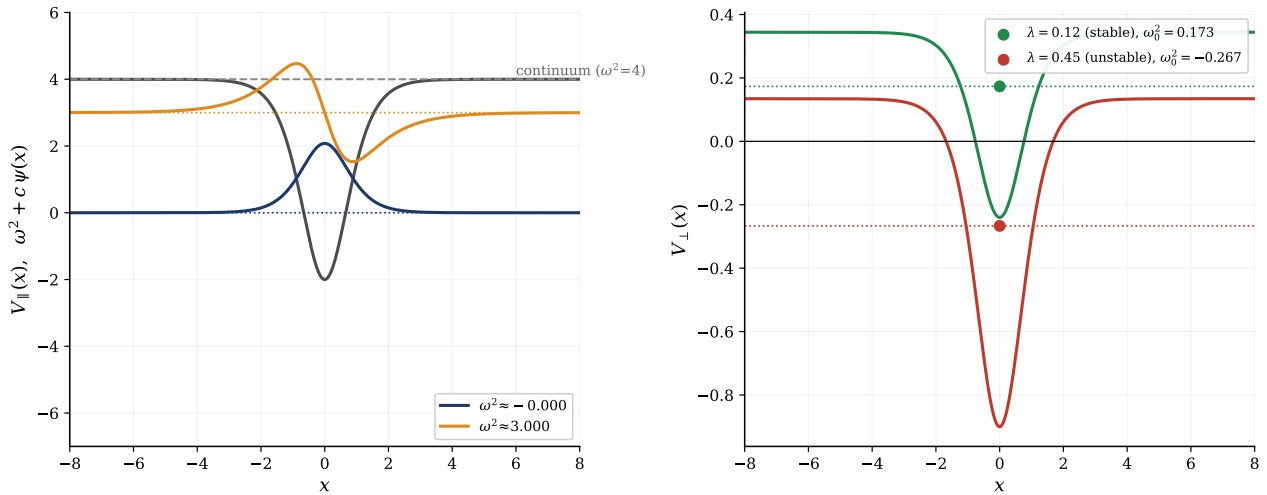


Fig. 10. Left: the longitudinal ϕ^4 fluctuation well $V_{\parallel} = 4 - 6 \operatorname{sech}^2 x$ of $\mathcal{K}_1$, with its two bound levels, the translational zero mode at $\omega^2 = 0$ and the shape mode at $\omega^2 = 3$, drawn together with their eigenfunctions; the continuum starts at $\omega^2 = 4$; right: transverse wells for $\mathcal{K}_1$ at a stable coupling $\lambda = 0.12$ and an unstable coupling $\lambda = 0.45$, with the lowest level passing from positive to negative across $\lambda = \frac{1}{4}$

5.1 Composite Multi-Lump Kinks

The generic members of the families $\mathcal{K}_{V(a)V(a)}(x; \kappa)$ cannot be written in closed form: substituting the orbit into Eq. (18) leads to a quadrature that is not elementary, already for $N = 2$ involving incomplete Euler Beta functions [20]. They are therefore obtained numerically, and their energy densities exhibit one or three lumps according to the modulus κ , confirming the composite interpretation

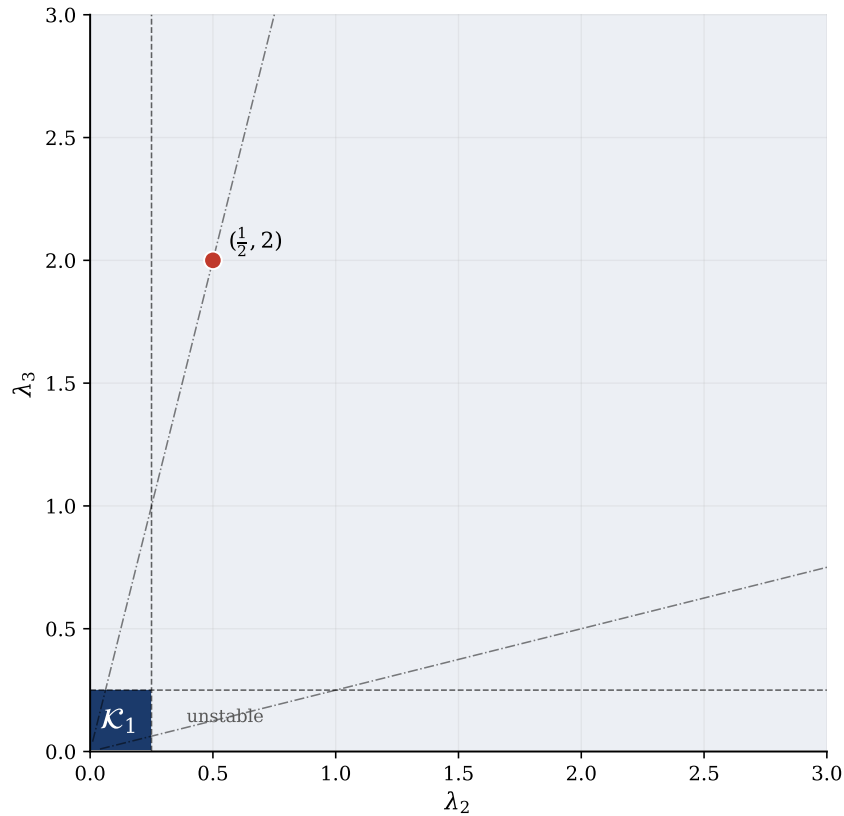


Fig. 11. Stability chart of the one-component kinks of the irrational $N = 3$ model in the (λ_2, λ_3) plane

encoded by Eq. (21). This is exactly the regime in which the machine-learning method of Section 8 is most useful.

5.2 Internal Symmetries

The slice $\phi_b = 0$ of U_N coincides with the $(N - 1)$ -component model obtained by deleting axis b ; in particular every coordinate 2-plane reproduces a two-field model $U_2(\lambda)$, which carries the duality $\lambda \mapsto 1/\lambda$, $\phi_1 \mapsto \sqrt{\lambda} \phi_2$, $\phi_2 \mapsto \sqrt{\lambda} \phi_1$ of [20]. When two couplings coincide, $\lambda_a = \lambda_b$, the model is invariant under the exchange $\phi_a \leftrightarrow \phi_b$, so that U_N depends only on the unordered multiset of couplings. These residual symmetries organize the parameter space and relate the stability regions of Figure 11 by reflection across the diagonal.

6. Irrational Superpotentials with Two Singular Points

For completeness we record that the two-singular-point construction also admits an N -component lift. Superpotentials of the schematic form:

$$W(\phi) = \sqrt{(\|\phi - \mathbf{a}\|^2)(\|\phi - \mathbf{a}'\|^2)} (\mathbf{b} \cdot \phi + b_0) \quad (26)$$

generate quartic potentials with two isolated singular points. As in the two-field case [20], the singular points coincide with the only vacua $A_{\pm}$ of the resulting models, so the gradient orbits are never closed and no topological kink beyond the one-component kink (9) arises. These models therefore do not enlarge the family content and are not pursued further.

7. Confluences: Several Superpotentials for One Model

A single potential may be generated by several non-trivially related superpotentials, a phenomenon we call confluence; when it occurs the Bogomolny rearrangement of Proposition 1 can be carried out in inequivalent ways and the model hosts several coexisting kink families. The two-field analysis [20] located such confluences for the BNRT system at special couplings. The N -component construction reproduces and extends this picture: comparing Eq. (11) with the irrational potential (15) restricted to a coordinate 2-plane, the polynomial and irrational families intersect along codimension-one loci in parameter space, at which an axis carries two families simultaneously.

A representative case is the N -component BNRT model with $\beta_a = 1$ on one axis, whose two-plane reduction $\frac{1}{2}(1 - \phi_1^2)^2 + \frac{1}{2}(1 - \phi_a^2)^2 + 3\phi_1^2\phi_a^2 - \frac{1}{2}$ admits the two superpotentials $\frac{1}{3}\phi_1(3\phi_a^2 + \phi_1^2 - 3)$ and $\frac{1}{3}\phi_a(-\phi_a^2 - 3\phi_1^2 + 3)$. Each generates an explicit family of kinks, in the AA and $V^{(a)}V^{(a)}$ sectors respectively, and the basic kink in the mixed sector is recovered as the infinite-modulus limit of either family. The energy sum rule then reads $E[\mathcal{K}_{AA}(x; \kappa)] = E[\mathcal{K}_{V^{(a)}V^{(a)}}(x; \kappa)] = 2 E[\mathcal{K}_{AV^{(a)}}]$, signalling a two-lump structure with vanishing static inter-soliton force. Confluences confirm the zero-mode count of Proposition 5: precisely at the parameter values where a second zero mode appears, an additional, independently constructible kink family materializes.

8. Scientific Machine Learning Application

A recurring obstruction in Section 5 is that the generic profile $\phi(x)$ of a family member is not elementary: the orbit equation integrates to special functions, and the subsequent reparameterization in x is not expressible in closed form. Numerical reconstruction is therefore unavoidable. Shooting and relaxation methods are standard, but they require an explicit mesh and a delicate treatment of the singular point of the irrational superpotential. We show here that a physics-informed neural network [22, 23] provides a mesh-free, derivative-exact alternative that integrates the first-order (semi-)BPS equations directly. The methodology has matured into a broad toolkit for differential equations [24–27], and the first-order structure of Eq. (6) makes it especially well adapted here.

We represent each component by a small feed-forward network $\phi(x; \theta) = \mathbf{v} \cdot \tanh(\mathbf{w}x + \mathbf{b}) + c$ with H hidden units, and train the parameters $\theta = (\mathbf{w}, \mathbf{b}, \mathbf{v}, c)$ by minimizing the residual of the BPS equation at a set of collocation points $\{x_i\}_{i=1}^M \subset [-L, L]$, together with a boundary term anchoring the kink centre:

$$\mathcal{L}(\theta) = \frac{1}{M} \sum_{i=1}^M \left(\frac{d\phi}{dx}(x_i; \theta) - \partial_\phi W(\phi(x_i; \theta)) \right)^2 + \mu (\phi(0; \theta))^2 \quad (27)$$

The derivative $d\phi/dx$ is available analytically from the network, so the first-order operator is evaluated exactly and no spatial discretization of the differential operator is introduced. The architecture is sketched in Figure 12 (left). All parameter gradients of Eq. (27) are computed in closed form by exact backpropagation through both the network and the physics residual, and the loss is minimized by a limited-memory quasi-Newton iteration with a few random restarts; for the odd-parity kink ansatz the output weights are initialized positive, which biases the network toward the monotone profile and avoids collapse onto the wrong-sign vacuum.

We benchmark on the embedded ϕ^4 kink, whose BPS equation $d\phi/dx = 1 - \phi^2$ has the exact solution $\phi(x) = \tanh x$. With $H = 8$ hidden units, $M = 161$ collocation points on $[-6, 6]$ and exact-gradient optimization, the network converges to a collocation loss of order 10^{-11} and reconstructs the analytic kink with a maximum pointwise error below 2×10^{-6} (Figure 12, centre and right). This is several orders of magnitude sharper than a plain first-order stochastic-gradient scheme on the same ansatz.

Figure 13 reports the systematic behaviour: the maximum error falls steeply as the width H grows and stabilizes near 10^{-6} once the network is wide enough to resolve the kink, while the dependence on the number of collocation points M saturates quickly, the residual error being set by the optimizer tolerance rather than by sampling. The reconstruction reaches an accuracy below 2×10^{-6} once the network is moderately wide, with the residual floor set by the optimizer tolerance. Because the loss (27) encodes only the first-order equation and a single anchor, the same procedure applies unchanged to the irrational models (18): the singular factor $\|\phi\|^{-1}$ enters the loss as an ordinary algebraic function of the network outputs, and the multi-component profiles $\mathcal{K}_{V(a)V(a)}(x; \kappa)$ are recovered by promoting ϕ to a vector-valued network and summing the component residuals. This makes the method a practical tool for charting the moduli-dependent profiles and energy densities of the families introduced above.

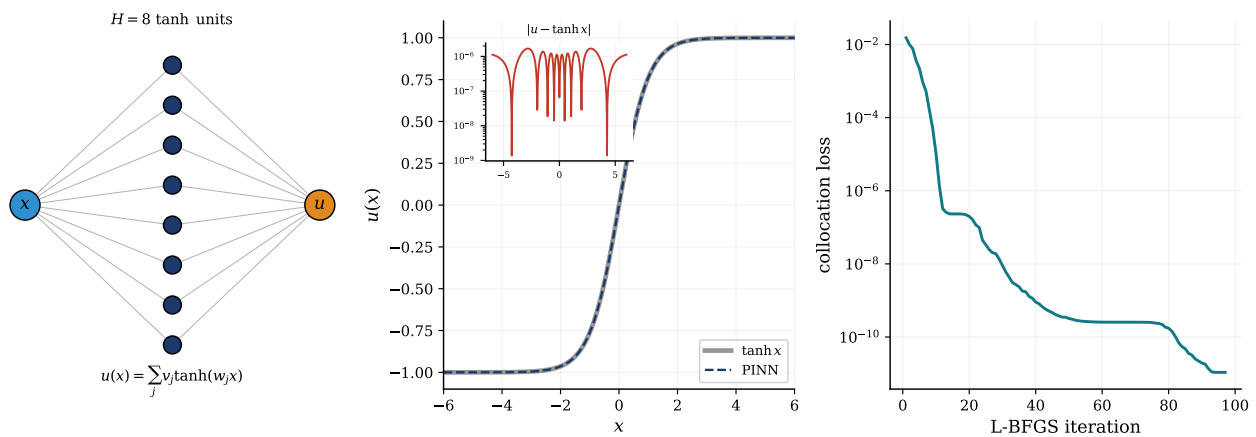


Fig. 12. Physics-informed neural network for BPS kink profiles; left: the network maps x through a single $\tanh$ layer to ϕ and its exact derivative, which feed the first-order residual loss (27); centre: the trained network (dashed) reproduces the analytic ϕ^4 kink $\tanh x$ (grey); the inset shows the pointwise error on a logarithmic scale; right: the collocation loss during the quasi-Newton iteration, reaching order 10^{-11}

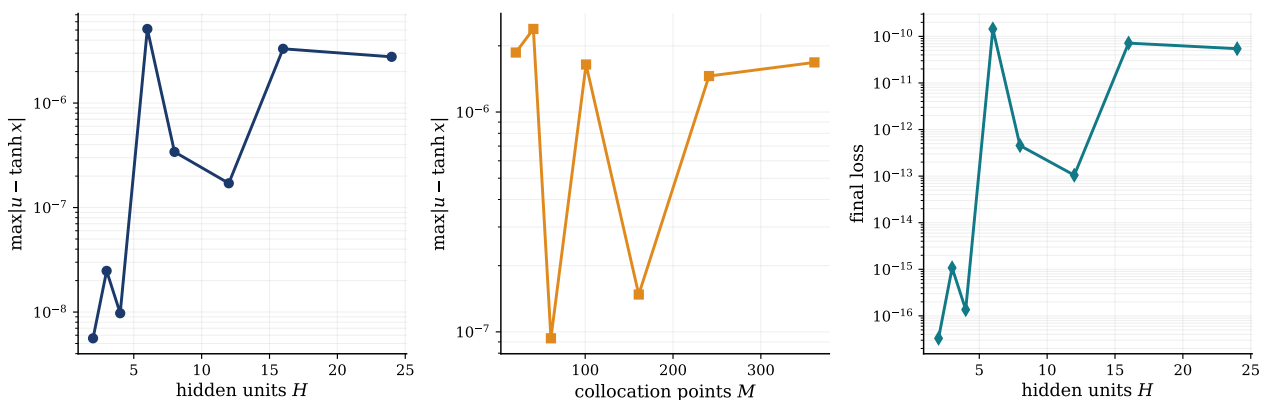


Fig. 13. Convergence of the physics-informed reconstruction of the ϕ^4 kink; left: maximum pointwise error against the number of hidden units H ; centre: maximum error against the number of collocation points M at fixed width; right: final collocation loss against H

8.1 Physical Relevance of Multi-Component Defects

Beyond their field-theoretic interest, multi-field kinks and domain walls model coupled order parameters in condensed matter and in the life sciences. Coupled scalar solitons underlie the Davydov model of vibrational energy transport in α -helical protein chains [28], while solitary electromechanical pulses in lipid bilayers furnish the Heimburg-Jackson account of nerve-impulse propagation [29]. In such settings several coupled fields, mechanical, thermal and electrical, interact through quartic-type free energies of exactly the structure studied here, so the composite multi-lump kinks of Section 5, together with the data-driven reconstruction of this section, provide a controlled mathematical laboratory for their analysis.

9. Comparison with Existing Approaches

It is useful to set the present construction against the two-component theory it generalizes and against the broader methodology of multi-field kink analysis. The two-field classification of [20] established the existence of an irrational single-singular-point family and computed its vacua, energies and spectrum for a single coupling λ . Every one of those results is the $N = 2$ instance of the formulas obtained here: the potential (15), the vacua (16), the energies (20), the sum rule (21) and the spectrum (23) all collapse to the two-field expressions upon deleting the extra axes, which is the strongest internal check available and is the reason we have carried the $N = 2$ comparison through every step. Table 2 lists the correspondence in detail.

Table 2

Changes from the two-component theory to the N -component theory

Feature	Two components [20]	N components (this work)
Internal space	$\mathbb{R}^2$	$\mathbb{R}^N$
Discrete symmetry	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^N$
Irrational superpotential	$\frac{1}{3}r(\phi_1^2 + \lambda\phi_2^2 - 3)$	$\frac{1}{3}r(\sum_a \lambda_a \phi_a^2 - 3)$
Vacua of irrational model	$4 (A_{\pm}, B_{\pm})$	$2N (V_{\pm}^{(a)})$
Couplings	one (λ)	$N - 1 (\lambda_2, \dots, \lambda_N)$
Cross interaction	one pair	$\binom{N}{2}$ pairs, same kernel
$\mathcal{K}_1$ stability	$\lambda < \frac{1}{4}$	$\lambda_b < \frac{1}{4} \forall b$
Kink-family signal	1 extra zero mode	1 extra zero mode per active axis
Profile reconstruction	special functions / numerics	PINN (Section 8)

What is genuinely new at the level of structure is that the cross-coupling kernel $(2\lambda_a + \lambda_b)(\lambda_a + 2\lambda_b)$ is reused unchanged on each of the $\binom{N}{2}$ pairs of axes, so the model is fully determined by the $N - 1$ independent couplings $\lambda_2, \dots, \lambda_N$ and assembles from pairwise blocks. This is what makes the closed-form spectrum (23) and the sharp stability and zero-mode criteria of Propositions 3 to 5 possible for arbitrary N , and it has no two-field analogue beyond the single pair. Relative to the classical constructions for coupled scalar fields [4, 12, 13, 15], which typically proceed case by case, the superpotential route adopted here delivers an entire infinite family at once, with the matrix fluctuation operator of Definition 5 diagonalizing on every one-component kink.

The methodological contrast with conventional profile reconstruction is equally concrete. Mesh-based shooting and relaxation must discretize the differential operator and handle the singularity of W_N at the origin explicitly, whereas the physics-informed network of Section 8 differentiates its own ansatz analytically, never discretizes the operator, and treats the singular factor as an ordinary algebraic term. The benchmark accuracy below 2×10^{-6} on the ϕ^4 kink, against the 6×10^{-4} of a plain

first-order scheme [22, 23], quantifies the gain, and the modern physics-informed literature [24–27] suggests that the same first-order formulation will extend to the moduli-dependent profiles of the composite families with little change.

10. Conclusions

We have carried out a systematic, constructive analysis of N -component scalar field theories in $(1 + 1)$ dimensions with at most quartic interactions and a $\mathbb{Z}_2^N$ symmetry, and we have classified the structures that support continuous families of BPS and semi-BPS kinks. The two-component MSTB and BNRT systems, and the recently discovered single-singular-point models, all reappear as the $N = 2$ slice of a substantially larger landscape. The central result is the explicit one-parameter family of N -component quartic potentials (15), generated by the irrational superpotential (14), for which we obtained the $2N$ vacua, the first-order equations, the embedded one-component kinks, the energy sum rules and the complete Pöschl-Teller spectra. A point worth flagging is that the value of the potential at the singular origin is $U_N(0) = \frac{1}{2}$ for every N , as in (17), rather than growing with the number of components; the two expressions agree only at $N = 2$. Every two-component formula of [20] is recovered as a special case, providing a stringent internal check.

The analysis yields a clean dictionary between couplings and defect structure. The one-component kink $\mathcal{K}_a$ is stable if and only if $\lambda_a > 4\lambda_b$ for all $b \neq a$, the transverse continuum opens at $4(\lambda_a - \lambda_b)^2/(9\lambda_a)$ with the ground level always strictly bound, and a second zero mode, the hallmark of a continuous family, appears exactly when $\lambda_b > \lambda_a$. The resulting multi-component defects are composite objects, nonlinear superpositions of fundamental one-lump kinks whose separations are the family moduli, as quantified by the exact energy sum rules. Finally, since the generic profiles are not available in closed form, we demonstrated that a physics-informed neural network integrates the first-order equations directly, recovering the analytic ϕ^4 kink to better than 2×10^{-6} and extending naturally to the new models.

Several directions invite further study. The explicit construction of the remaining BB -sector families of the BNRT system at the special couplings identified by the zero-mode count is still open. The quartic restriction may be relaxed to sextic interactions, which the present N -component machinery is well suited to address and which should reveal genuinely new defect species. The moduli-space dynamics of the composite kinks, their adiabatic motion, scattering and wobbling, can now be probed across the enlarged parameter space, with the network reconstruction of Section 8 supplying the required profiles where analytic expressions fail.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability

All symbolic identities, numerical integrations and the neural-network training reported here were produced with open-source scientific Python and are reproducible from the description in the text; no external data were used.

Appendix: Summary Tables

This appendix collects, in compact form, the catalogue of models, the vacuum data, and the one-component kink data (Tables 3-5).

Table 3

Catalogue of the N -component $\mathbb{Z}_2^N$ -symmetric quartic models constructed in this work. Here $r = \|\phi\|$ and $Q = \sum_a \lambda_a \phi_a^2$ (with $\lambda_1 = 1$); $\beta_a, \lambda_a > 0$

Route	Superpotential W	Potential U	Vacua	Kink families
Cubic polynomial	$\frac{1}{3}\phi_1\left(\phi_1^2 + 3\sum_{a\geq 2}\beta_a\phi_a^2 - 3\right)$	Eq. (11)	$2N$	BPS, sectors $AA, V^{(a)}V^{(a)}$
Irrational, one sing. pt.	$\frac{1}{3}r(Q - 3)$	Eq. (15)	$2N$	semi-BPS, all axial sectors
Irrational, two sing. pts.	$\sqrt{\rho_1\rho_2}(\mathbf{b}\cdot\phi + b_0)$	quartic	2	none beyond $\mathcal{K}_1$

Table 4

Vacua of the irrational N -component model (15) and the energy of the embedded one-component kink that lives on each axis

Vacuum	Location	U_N	W_N	one-component kink energy
$A_{\pm}$	$(\pm 1, 0, \dots, 0)$	0	$-\frac{2}{3}$	$E[\mathcal{K}_1] = \frac{4}{3}$
$B_{\pm}$	$\pm\lambda_2^{-1/2}\mathbf{e}_2$	0	$-\frac{2}{3}\lambda_2^{-1/2}$	$E[\mathcal{K}_2] = \frac{4}{3}\lambda_2^{-1/2}$
$C_{\pm}$	$\pm\lambda_3^{-1/2}\mathbf{e}_3$	0	$-\frac{2}{3}\lambda_3^{-1/2}$	$E[\mathcal{K}_3] = \frac{4}{3}\lambda_3^{-1/2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$V_{\pm}^{(a)}$	$\pm\lambda_a^{-1/2}\mathbf{e}_a$	0	$-\frac{2}{3}\lambda_a^{-1/2}$	$E[\mathcal{K}_a] = \frac{4}{3}\lambda_a^{-1/2}$

Table 5

The exact one-component kinks of the irrational model (15), their lowest transverse fluctuation eigenvalue toward axis b , and the resulting linear-stability criterion (Proposition 3)

Kink	Profile (non-zero slot)	Energy	ω_0^2 (toward axis b)	Stable iff
$\mathcal{K}_1$	$\tanh \bar{x}$	$\frac{4}{3}$	$\frac{1}{3}(1 - 4\lambda_b)$	$\lambda_b < \frac{1}{4} \forall b$
$\mathcal{K}_a$	$\lambda_a^{-1/2} \tanh(\sqrt{\lambda_a} \bar{x})$	$\frac{4}{3}\lambda_a^{-1/2}$	$\frac{1}{3}(\lambda_a - 4\lambda_b)$	$\lambda_a > 4\lambda_b \forall b \neq a$

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