



# Generalized Enriched Cyclic Contractions in Controlled Metric Spaces: Fixed Point Results, Iterated Function Systems, and a Physics-Informed Neural Network Realization

Sahib Yar<sup>1,\*</sup>, Khaleel Ahmad<sup>2,3</sup>, Nida Najeeb<sup>1</sup>

<sup>1</sup> Department of Mathematics and Statistics, University of Southern Punjab, Multan 60000, Pakistan

<sup>2</sup> Department of Mathematics and Statistics, The University of Lahore, Pakistan

<sup>3</sup> Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku, AZ1096, Azerbaijan

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## ABSTRACT

This research introduces the class of generalized enriched cyclic contractions in the setting of controlled metric spaces equipped with a convex structure. A unified fixed point theory for this class is developed. Enrichment, cyclicity, and point-dependent controlled distances have not been treated together. Besides, no single principle recovered the classical results while also supporting fractal and neural applications. By combining the Krasnoselskij averaging device with a cyclic representation of the underlying space, we prove existence, uniqueness, and convergence theorems together with explicit a priori and a posteriori error estimates. Also, we recover Banach, Kannan, Reich, and Edelstein type principles as particular cases. As an application, we construct a generalized iterated function system driven by a finite family of enriched cyclic contractions. We study the associated Hutchinson operator on the hyperspace of nonempty compact subsets endowed with the controlled Pompeiu–Hausdorff metric. We further give a physics-informed neural network realization. The solution of an initial value problem is identified with the unique fixed point of a Picard operator that is a contraction in a Bielecki weighted metric. The framework strictly enlarges several results in the literature, connecting fixed point theory, controlled metric geometry, fractal analysis, and scientific machine learning.

\*Corresponding author.

E-mail address: [sahibyar770@gmail.com](mailto:sahibyar770@gmail.com)

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## 1. Introduction

The metric fixed point theory originates from the celebrated principle of Banach [1], which guarantees that every contraction of a complete metric space into itself possesses a unique fixed point that can be reached by successive approximations. The strength of this principle lies in its constructive nature and in the geometric decay of the approximation error, yet its scope is constrained by the rigidity of the contraction requirement. A long line of research has therefore been devoted to relaxing the contractive hypothesis. Kannan [2] discovered a contractive condition that does not presuppose continuity of the operator, Chatterjea [3] proposed a symmetric counterpart, while Reich [4], Hardy and Rogers [5], and Ćirić [6] unified and extended these ideas into broad families of generalized contractions that subsume the Banach, Kannan, and Chatterjea principles simultaneously. In a complementary direction, Edelstein [7] showed that on a compact metric space the strict contractive inequality already forces the existence of a unique fixed point, an observation that remains indispensable whenever a global contraction constant is unavailable.

A second major theme, closely interwoven with the present work, is the theory of cyclic contractions, which was systematically initiated by Kirk, Srinivasan, and Veeramani [8]. Their idea of imposing a contractive condition only between consecutive members of a cyclic decomposition of the space, rather than on the whole space, proved remarkably fruitful and opened a new avenue in fixed point theory. Rus [9] introduced the notion of a cyclic representation and demonstrated its flexibility, and subsequent contributions by Păcurar & Rus [10], Petric [11], Karapınar [12], Karapınar & Sadarangani [13], Abbas *et al.* [14], and Jleli & Samet [15] produced a wealth of fixed point and best proximity results under various cyclic contractive conditions. These developments revealed that cyclic techniques provide a versatile framework for extending classical fixed point principles and carry concrete applications in approximation theory and nonlinear analysis.

A third and more recent ingredient is the enrichment technique based on Krasnoselskij averaging. Berinde & Păcurar [16–18] observed that many operators which are not themselves contractive become contractive after averaging with the identity, that is, after passing from  $T$  to the averaged operator  $T_\lambda = (1 - \lambda)I + \lambda T$ . This single device enlarges the applicable class of mappings dramatically while preserving the fixed point set, and it has reinvigorated the study of enriched contractions, enriched nonexpansive mappings, and their cyclic variants. In order to carry the averaging construction beyond normed spaces, one relies on the convex structure introduced by Takahashi [19], which axiomatizes the geometric notion of a convex combination in an abstract metric setting and makes the averaged operator meaningful in spaces with no linear structure. The synthesis of cyclicity and enrichment was carried out by Abbas *et al.* [20], whose generalized enriched cyclic contractions and the attendant iterated function systems constitute the principal point of departure for the present paper.

In parallel, the metric structures themselves have been progressively generalized. The  $b$ -metric spaces of Bakhtin [21] and Czerwik [22] weakened the triangle inequality by a fixed multiplicative constant, and Mlaiki *et al.* [23] introduced controlled metric type spaces, in which the triangle inequality is moderated by a control function  $\omega : X \times X \rightarrow [1, \infty)$  that may depend on the points involved. This point-dependent relaxation accommodates a far wider range of distance structures, and it has been further developed in the double-controlled and extended settings by Abdeljawad *et al.* [24], Asim *et al.* [25], Mlaiki [26], Karapınar *et al.* [27], Shatanawi *et al.* [28], and Souayah *et al.* [29]. A recurring subtlety in this circle of ideas is that the controlled distance need not be continuous and that the naive iteration of the controlled triangle inequality may fail to produce a convergent telescopic estimate; both phenomena must be handled with care, and they are addressed explicitly in the analysis below.

A fourth strand concerns iterated function systems and fractal geometry. Hutchinson [30] proved

that a finite family of contraction mappings determines a unique nonempty compact attractor, and Barnsley [31] developed the theory into a powerful instrument for fractal modeling and image synthesis. The set-valued analysis underlying these constructions rests on the Pompeiu–Hausdorff metric and on the foundational work of Nadler [32] on multivalued contractions. More recently the classical contractivity hypothesis has been relaxed in the generalized iterated function systems studied by Miculescu & Mihail [33], Georgescu *et al.* [34], and Strobin [35], where attractors persist under substantially weaker conditions. Embedding such systems in controlled metric spaces, and driving them by enriched cyclic contractions, is one of the contributions of the present work.

Finally, fixed point theory has acquired a striking computational counterpart through scientific machine learning. The physics-informed neural networks of Raissi *et al.* [36] approximate the solution of a differential equation by training a network to minimize the equation residual together with the initial or boundary data. When the differential problem is recast in integral form, its solution is precisely the fixed point of a Picard type operator, and the classical observation of Bielecki [37] that an exponentially weighted norm renders this operator a contraction provides the missing quantitative bridge between the abstract theory and the trained surrogate. Since every metric is a controlled metric with control function identically equal to one, the Bielecki contraction falls within the scope of our framework, and the averaged Picard operator becomes an enriched contraction with the same fixed point.

The preceding survey exposes a clear research gap. The enriched cyclic contractions of [20] were developed in standard metric spaces, the controlled metric type spaces of [23] were not equipped with a convex structure suitable for averaging, and the iterated function systems of [30, 31] were not connected to enrichment or to cyclicity. No existing result simultaneously (i) admits a point-dependent controlled triangle inequality, (ii) permits the Krasnoselskij averaging of a possibly non-contractive operator through an abstract convex structure, (iii) imposes the contractive condition only cyclically and in the generalized Reich form that contains the Banach and Kannan conditions as special cases, and (iv) yields, as a by-product, both a fractal attractor theory and a quantitatively controlled neural approximation. The aim of this paper is to fill exactly this gap. We introduce generalized enriched cyclic contractions in convex controlled metric spaces, we prove existence, uniqueness, convergence, and explicit error estimates, and we recover the Banach, Kannan, Reich, and Edelstein principles as corollaries. We then build a generalized iterated function system from a finite family of such contractions, we analyze the Hutchinson operator on the hyperspace endowed with the controlled Pompeiu–Hausdorff metric, and we establish the existence and uniqueness of attractors. We close by realizing the theory computationally through a physics-informed neural network whose measured accuracy matches the contraction estimates. The logical architecture of the paper is displayed in Figure 1, and the surfaces and attractors that accompany the analysis are collected in Figure 3. Each numerical figure is drawn directly from the model it depicts, so the reported rates and the plotted curves refer to the same computation.

## 2. Preliminaries

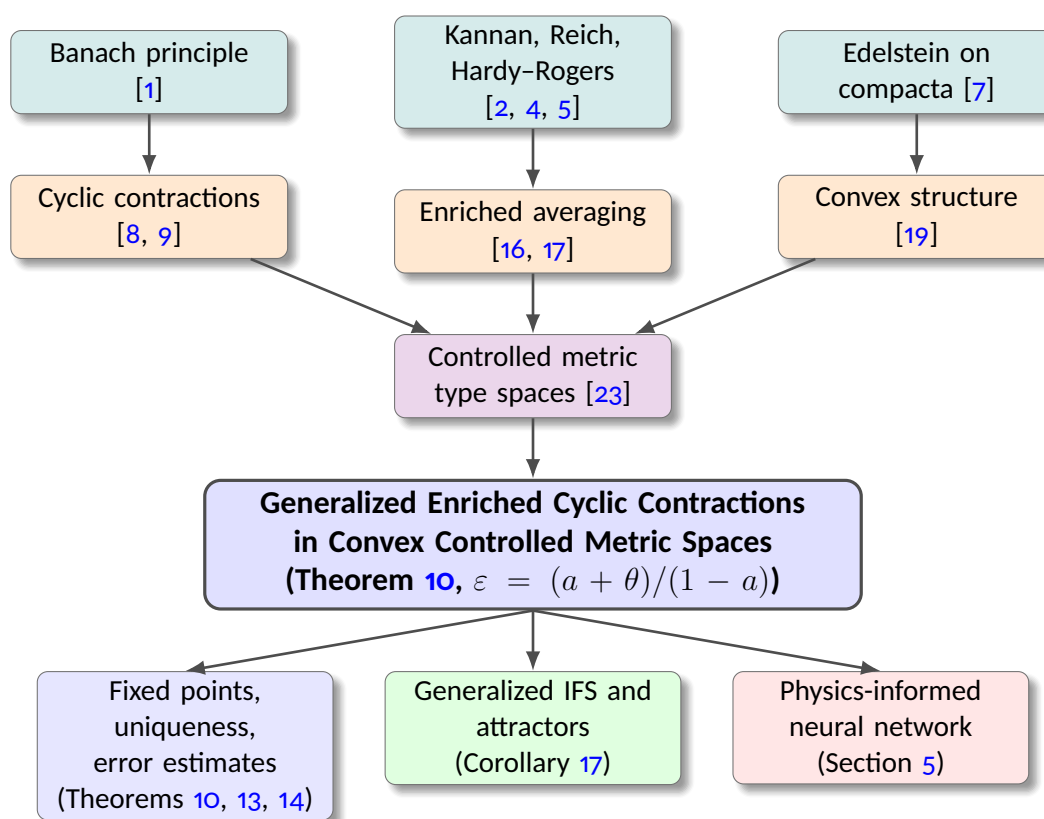
Throughout this section  $X$  denotes a nonempty set. We recall the notions required in the sequel and fix the notation. The controlled metric type spaces below are due to Mlaiki, Aydi, Souayah, and Abdeljawad [23].

**Definition 1.** Let  $\omega : X \times X \rightarrow [1, \infty)$  be a function and let  $d_\omega : X \times X \rightarrow [0, \infty)$  satisfy, for all  $x, y, z \in X$ ,

1.  $d_\omega(x, y) = 0$  if and only if  $x = y$ ;
2.  $d_\omega(x, y) = d_\omega(y, x)$ ;

$$3. d_{\omega}(x, z) \leq \omega(x, z)[d_{\omega}(x, y) + d_{\omega}(y, z)].$$

Then the pair  $(X, d_{\omega})$  is called a controlled metric space with control function  $\omega$ .



**Fig. 1.** Logical architecture of the paper: classical principles are enriched, made cyclic, and transplanted into controlled metric spaces to yield a single framework with three applications

Every metric space is a controlled metric space with  $\omega \equiv 1$ , and every  $b$ -metric space with constant  $s \geq 1$  in the sense of Bakhtin [21] and Czerwik [22] is a controlled metric space with  $\omega \equiv s$ .

**Definition 2.** A sequence  $\{x_n\}$  in  $(X, d_{\omega})$  converges to  $x \in X$  if  $\lim_{n \rightarrow \infty} d_{\omega}(x_n, x) = 0$ . The sequence is Cauchy if  $\lim_{m, n \rightarrow \infty} d_{\omega}(x_n, x_m) = 0$ . The space  $(X, d_{\omega})$  is complete if every Cauchy sequence converges to a point of  $X$ . The controlled distance  $d_{\omega}$  is continuous if  $d_{\omega}(x_n, y_n) \rightarrow d_{\omega}(x, y)$  whenever  $x_n \rightarrow x$  and  $y_n \rightarrow y$ .

In a general controlled metric space the distance  $d_{\omega}$  need not be continuous, so continuity, when required, will be assumed explicitly. The convex structure recalled next is due to Takahashi [19]; it transplants the notion of a convex combination to a setting without linear operations.

**Definition 3.** A controlled metric space  $(X, d_{\omega})$  endowed with a mapping  $W : X \times X \times [0, 1] \rightarrow X$  is called a convex controlled metric space if, for all  $u, x, y \in X$  and all  $\lambda \in [0, 1]$ ,

$$d_{\omega}(u, W(x, y, \lambda)) \leq (1 - \lambda)d_{\omega}(u, x) + \lambda d_{\omega}(u, y). \quad (1)$$

In a normed space the choice  $W(x, y, \lambda) = (1 - \lambda)x + \lambda y$  satisfies Eq. (1).

The cyclic representation that organizes the contractive condition is due to Kirk *et al.* [8] and Rus [9].

**Definition 4.** Let  $Y = \bigcup_{j=1}^p A_j$ , where  $A_1, \dots, A_p$  are nonempty subsets of  $X$ . A mapping  $S : Y \rightarrow Y$  admits a cyclic representation with respect to  $\{A_j\}$  if  $S(A_j) \subseteq A_{j+1}$  for  $j = 1, \dots, p$ , with the convention  $A_{p+1} = A_1$ .

Following the enrichment philosophy of Berinde & Păcurar [16, 17], we average a given operator with the identity through the convex structure.

**Definition 5.** Let  $T : Y \rightarrow X$  and let  $\lambda \in (0, 1]$ . The averaged mapping associated with  $T$  is:

$$T_\lambda x = W(x, Tx, \lambda), \quad x \in Y. \quad (2)$$

In a normed space this reduces to the Krasnoselskij operator  $T_\lambda x = (1 - \lambda)x + \lambda Tx$ , and  $\text{Fix}(T) = \text{Fix}(T_\lambda)$ .

The central contractive class of the paper combines the cyclic, enriched, and generalized Reich features.

**Definition 6.** Let  $Y = \bigcup_{j=1}^p A_j$ . A mapping  $T : Y \rightarrow X$  is called a generalized enriched cyclic contraction if  $\{A_j\}$  is a cyclic representation of  $Y$  with respect to  $T_\lambda$  and there exist constants  $a, \theta \in [0, 1)$  such that, for all  $x \in A_j$  and  $y \in A_{j+1}$ ,

$$d_\omega(T_\lambda x, T_\lambda y) \leq \theta d_\omega(x, y) + a[d_\omega(x, T_\lambda x) + d_\omega(y, T_\lambda y)]. \quad (3)$$

The choice  $a = 0$  yields an enriched cyclic Banach contraction, while  $\theta = 0$  yields an enriched cyclic Kannan contraction.

For the iterated function system part we work in the hyperspace of compact sets. For a nonempty subset  $A \subseteq X$ , let  $C(A)$  denote the family of all nonempty compact subsets of  $A$ . The following Pompeiu–Hausdorff construction adapts Nadler [32] to the controlled setting.

**Definition 7.** For  $A, B \in C(X)$  put

$$d_\omega(a, B) = \inf_{b \in B} d_\omega(a, b), \quad D_\omega(A, B) = \sup_{a \in A} d_\omega(a, B),$$

and define the controlled Pompeiu–Hausdorff metric by

$$H_\omega(A, B) = \max\{D_\omega(A, B), D_\omega(B, A)\}.$$

If  $T : X \rightarrow X$  maps compact sets to compact sets, the induced set mapping  $T^* : C(X) \rightarrow C(X)$  is  $T^*(A) = T(A)$ . A generalized iterated function system is a finite family  $\mathcal{T} = \{T^{(1)}, \dots, T^{(N)}\}$  of generalized enriched cyclic contractions, and its associated Hutchinson operator  $F : C(X) \rightarrow C(X)$  is

$$F(B) = \bigcup_{n=1}^N (T_\lambda^{(n)})^*(B), \quad B \in C(X). \quad (4)$$

A set  $A^* \in C(X)$  with  $F(A^*) = A^*$  is called an attractor of the system. The two lemmas below are the controlled analogues of the classical estimates of Hutchinson [30] and Barnsley [31].

**Lemma 8.** Let  $(X, d_\omega)$  be a controlled metric space and let  $\{C_i\}_{i=1}^N$  and  $\{D_i\}_{i=1}^N$  be finite collections of nonempty compact subsets of  $X$ . Then

$$H_\omega\left(\bigcup_{i=1}^N C_i, \bigcup_{i=1}^N D_i\right) \leq \max_{1 \leq i \leq N} H_\omega(C_i, D_i).$$

**Proof.** Fix  $a \in \bigcup_i C_i$ , so  $a \in C_{i_0}$  for some  $i_0$ . Then

$$d_\omega\left(a, \bigcup_i D_i\right) = \inf_{b \in \bigcup_i D_i} d_\omega(a, b) \leq \inf_{b \in D_{i_0}} d_\omega(a, b) = d_\omega(a, D_{i_0}) \leq D_\omega(C_{i_0}, D_{i_0}) \leq \max_i H_\omega(C_i, D_i).$$

Taking the supremum over  $a \in \bigcup_i C_i$  gives  $D_\omega(\bigcup_i C_i, \bigcup_i D_i) \leq \max_i H_\omega(C_i, D_i)$ . The symmetric inequality is identical, and taking the maximum of the two yields the claim.  $\square$

**Lemma 9.** *If  $(X, d_\omega)$  is complete and  $A \subseteq X$  is closed, then  $C(A)$  is a closed subset of  $(C(X), H_\omega)$ .*

**Proof.** Let  $\{B_k\} \subseteq C(A)$  with  $B_k \rightarrow B$  in  $(C(X), H_\omega)$ . For each  $b \in B$  we have  $d_\omega(b, B_k) \leq D_\omega(B, B_k) \leq H_\omega(B, B_k) \rightarrow 0$ , so there exist  $b_k \in B_k \subseteq A$  with  $d_\omega(b, b_k) \rightarrow 0$ , hence  $b_k \rightarrow b$ . Since  $A$  is closed,  $b \in A$ . Thus  $B \subseteq A$  and  $B \in C(A)$ , so  $C(A)$  is closed.  $\square$

### 3. Main Results

We begin with the principal fixed point theorem for generalized enriched cyclic contractions. Two structural hypotheses accompany the contractive inequality. First, in accordance with Definition 2, we assume that the controlled distance is continuous, which is what legitimizes passage to the limit in the contractive inequality. Second, because iterating the controlled triangle inequality of Definition 1 produces products of control values whose convergence is delicate, we adopt the regularity device (Eq. (7)), namely the existence of a constant  $M \geq 1$  for which the distance between two iterates is dominated by  $M$  times the sum of the consecutive step lengths. This device holds with  $M = 1$  in every metric space, by the triangle inequality, and it isolates exactly the quantity needed to obtain a telescopic Cauchy estimate. The contractive inequality (5) is the generalized Reich condition of Definition 6. It contains the Banach condition for  $a = 0$  and the Kannan condition for  $\theta = 0$ .

**Theorem 10.** *Let  $(X, d_\omega)$  be a complete controlled metric space with control function  $\omega : X \times X \rightarrow [1, \infty)$  for which  $d_\omega$  is continuous, and let  $X$  be endowed with a convex structure  $W : X \times X \times [0, 1] \rightarrow X$ . Let  $A_1, \dots, A_p$  be nonempty closed subsets of  $X$ , put  $Y = \bigcup_{j=1}^p A_j$ , and assume that  $Y$  is complete. Let  $T : Y \rightarrow X$  and, for a fixed  $\lambda \in (0, 1]$ , let  $T_\lambda$  be the averaged mapping (2). Suppose that  $\{A_j\}$  is a cyclic representation of  $Y$  with respect to  $T_\lambda$ , and that there exist  $a, \theta \in [0, 1)$  such that, for all  $x \in A_j$  and  $y \in A_{j+1}$ :*

$$d_\omega(T_\lambda x, T_\lambda y) \leq \theta d_\omega(x, y) + a[d_\omega(x, T_\lambda x) + d_\omega(y, T_\lambda y)]. \quad (5)$$

Set

$$\varepsilon = \frac{a + \theta}{1 - a}$$

and assume that  $\varepsilon \in [0, 1)$ . Let  $\{x_n\}$  be the iterative sequence

$$x_{n+1} = T_\lambda x_n = W(x_n, T x_n, \lambda), \quad n \geq 0, \quad (6)$$

and suppose there exists  $M \geq 1$  such that, for all  $m > n$ ,

$$d_\omega(x_n, x_m) \leq M \sum_{k=n}^{m-1} d_\omega(x_k, x_{k+1}). \quad (7)$$

Then  $\{x_n\}$  converges to a point  $x^* \in \bigcap_{j=1}^p A_j$ , this point is the unique fixed point of  $T_\lambda$ , and the following estimates hold:

$$d_\omega(x_n, x^*) \leq M \frac{\varepsilon^n}{1 - \varepsilon} d_\omega(x_0, x_1), \quad n \geq 0, \quad (8)$$

$$d_{\omega}(x_{n+1}, x^*) \leq M \frac{\varepsilon}{1 - \varepsilon} d_{\omega}(x_n, x_{n+1}), \quad n \geq 0. \quad (9)$$

If, in addition,  $\text{Fix}(T) = \text{Fix}(T_{\lambda})$ , then  $T$  has a unique fixed point in  $Y$ .

*Proof.* Let  $x_0 \in Y$ , say  $x_0 \in A_j$ . Since  $\{A_j\}$  is cyclic with respect to  $T_{\lambda}$ , the iterate  $x_n$  belongs to  $A_{j+n}$  for every  $n$ , with indices read modulo  $p$ ; in particular consecutive iterates lie in consecutive sets, so the inequality (5) applies to the pair  $(x_n, x_{n+1})$ . Taking  $x = x_n$  and  $y = x_{n+1}$  and using  $T_{\lambda}x_n = x_{n+1}$  and  $T_{\lambda}x_{n+1} = x_{n+2}$ ,

$$d_{\omega}(x_{n+1}, x_{n+2}) \leq \theta d_{\omega}(x_n, x_{n+1}) + a[d_{\omega}(x_n, x_{n+1}) + d_{\omega}(x_{n+1}, x_{n+2})].$$

Because  $a \in [0, 1)$  the coefficient  $1 - a$  is strictly positive, so the term  $d_{\omega}(x_{n+1}, x_{n+2})$  may be collected on the left:

$$(1 - a) d_{\omega}(x_{n+1}, x_{n+2}) \leq (a + \theta) d_{\omega}(x_n, x_{n+1}).$$

Dividing by  $1 - a > 0$  produces the one-step contraction

$$d_{\omega}(x_{n+1}, x_{n+2}) \leq \varepsilon d_{\omega}(x_n, x_{n+1}), \quad \varepsilon = \frac{a + \theta}{1 - a}. \quad (10)$$

The standing assumption  $\varepsilon \in [0, 1)$  is equivalent to the sharp parameter restriction

$$\frac{a + \theta}{1 - a} < 1 \iff a + \theta < 1 - a \iff \theta < 1 - 2a, \quad (11)$$

which records exactly how the enriched Reich constants must be balanced. We now show by induction on  $n$  that

$$d_{\omega}(x_n, x_{n+1}) \leq \varepsilon^n d_{\omega}(x_0, x_1), \quad n \geq 0. \quad (12)$$

For  $n = 0$  the inequality is an identity. Assuming it at index  $n$ , the one-step estimate (10) gives

$$d_{\omega}(x_{n+1}, x_{n+2}) \leq \varepsilon d_{\omega}(x_n, x_{n+1}) \leq \varepsilon \cdot \varepsilon^n d_{\omega}(x_0, x_1) = \varepsilon^{n+1} d_{\omega}(x_0, x_1),$$

which is the inequality at index  $n + 1$ ; hence (12) holds for every  $n$ . Fix now  $m > n$ . The regularity device (7), the bound (12), and the finite geometric sum  $\sum_{k=n}^{m-1} \varepsilon^k = \varepsilon^n \frac{1 - \varepsilon^{m-n}}{1 - \varepsilon}$  yield

$$\begin{aligned} d_{\omega}(x_n, x_m) &\leq M \sum_{k=n}^{m-1} d_{\omega}(x_k, x_{k+1}) \leq M d_{\omega}(x_0, x_1) \sum_{k=n}^{m-1} \varepsilon^k \\ &= M \frac{\varepsilon^n (1 - \varepsilon^{m-n})}{1 - \varepsilon} d_{\omega}(x_0, x_1) \leq M \frac{\varepsilon^n}{1 - \varepsilon} d_{\omega}(x_0, x_1), \end{aligned} \quad (13)$$

where the last step used  $1 - \varepsilon^{m-n} \leq 1$ . Because  $\varepsilon \in [0, 1)$  the factor  $\varepsilon^n \rightarrow 0$ , so the right-hand side of Eq. (13) tends to zero as  $n \rightarrow \infty$  uniformly in  $m > n$ ; consequently  $\{x_n\}$  is a Cauchy sequence. Completeness of  $Y$  furnishes a limit  $x^* \in Y$  with  $x_n \rightarrow x^*$ . Holding  $n$  fixed and letting  $m \rightarrow \infty$  in Eq. (13), the continuity of  $d_{\omega}$  in its second argument (Definition 2) gives  $d_{\omega}(x_n, x_m) \rightarrow d_{\omega}(x_n, x^*)$ , and therefore the a priori estimate (8):

$$d_{\omega}(x_n, x^*) \leq M \frac{\varepsilon^n}{1 - \varepsilon} d_{\omega}(x_0, x_1), \quad n \geq 0.$$

Since the cyclic sequence visits each  $A_j$  infinitely often and each  $A_j$  is closed, the limit  $x^*$  belongs to every  $A_j$ , that is,  $x^* \in \bigcap_{j=1}^p A_j$ . To prove that  $x^*$  is fixed, apply Eq. (5) with  $x = x_n \in A_{j+n}$  and  $y = x^* \in A_{j+n+1}$ , which is permissible because  $x^*$  lies in every  $A_i$ :

$$d_{\omega}(x_{n+1}, T_{\lambda}x^*) = d_{\omega}(T_{\lambda}x_n, T_{\lambda}x^*) \leq \theta d_{\omega}(x_n, x^*) + a[d_{\omega}(x_n, x_{n+1}) + d_{\omega}(x^*, T_{\lambda}x^*)].$$

As  $n \rightarrow \infty$  we have  $d_\omega(x_n, x^*) \rightarrow 0$  and  $d_\omega(x_n, x_{n+1}) \rightarrow 0$ , while continuity of  $d_\omega$  and  $x_{n+1} \rightarrow x^*$  give  $d_\omega(x_{n+1}, T_\lambda x^*) \rightarrow d_\omega(x^*, T_\lambda x^*)$ . Hence

$$d_\omega(x^*, T_\lambda x^*) \leq a d_\omega(x^*, T_\lambda x^*),$$

and since  $a < 1$  we conclude  $d_\omega(x^*, T_\lambda x^*) = 0$ , that is,  $T_\lambda x^* = x^*$ .

For uniqueness, suppose  $u^*$  is another fixed point of  $T_\lambda$ . A fixed point of  $T_\lambda$  that is approached cyclically lies in  $\bigcap_j A_j$  by the argument just given, so Eq. (5) applies to the pair  $(x^*, u^*)$ :

$$d_\omega(x^*, u^*) = d_\omega(T_\lambda x^*, T_\lambda u^*) \leq \theta d_\omega(x^*, u^*) + a[d_\omega(x^*, T_\lambda x^*) + d_\omega(u^*, T_\lambda u^*)] = \theta d_\omega(x^*, u^*).$$

Since  $\theta < 1$ ,  $d_\omega(x^*, u^*) = 0$  and  $x^* = u^*$ .

It remains to establish the a posteriori estimate Eq. (9), which controls the error by the length of the most recent step. Iterating the one-step contraction Eq. (10) from any index  $k \geq n$  downward to  $n$  gives  $d_\omega(x_k, x_{k+1}) \leq \varepsilon^{k-n} d_\omega(x_n, x_{n+1})$ . Applying the regularity device (Eq. (7)) to the tail that starts at  $n+1$  and summing the resulting geometric series, for every  $m > n+1$ :

$$\begin{aligned} d_\omega(x_{n+1}, x_m) &\leq M \sum_{k=n+1}^{m-1} d_\omega(x_k, x_{k+1}) \leq M d_\omega(x_n, x_{n+1}) \sum_{k=n+1}^{m-1} \varepsilon^{k-n} \\ &= M d_\omega(x_n, x_{n+1}) \sum_{j=1}^{m-1-n} \varepsilon^j \leq M \frac{\varepsilon}{1-\varepsilon} d_\omega(x_n, x_{n+1}), \end{aligned}$$

where the final inequality uses  $\sum_{j=1}^{\infty} \varepsilon^j = \varepsilon/(1-\varepsilon)$ . Letting  $m \rightarrow \infty$  and using continuity of  $d_\omega$  in its second argument yields:

$$d_\omega(x_{n+1}, x^*) \leq M \frac{\varepsilon}{1-\varepsilon} d_\omega(x_n, x_{n+1}),$$

which is (9). Finally, if  $\text{Fix}(T) = \text{Fix}(T_\lambda)$  then  $Tx^* = x^*$ , and uniqueness for  $T$  follows from uniqueness for  $T_\lambda$ .  $\square$

The averaged map of Theorem 10 contracts the consecutive step lengths by the factor  $\varepsilon$ , and the a priori bound (8) governs the whole orbit. The role of the averaging parameter  $\lambda$  is isolated in Figure 2: a map that is merely nonexpansive becomes contractive once averaged, and smaller  $\lambda$  yields faster decay of  $|x_n - x^*|$  for the linear model  $x_{n+1} = (1-\lambda)x_n + \lambda k x_n$  with  $k = -0.8$ .

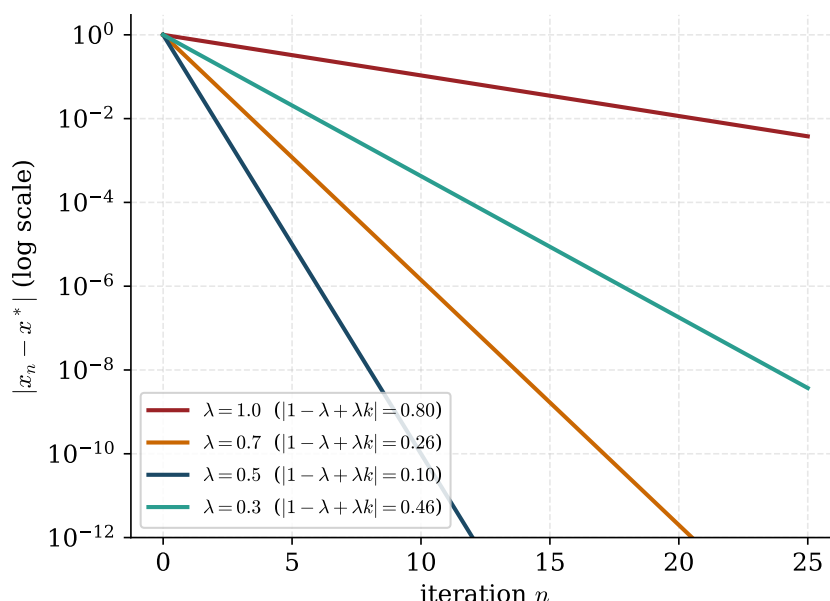
Taking  $a = 0$  in Theorem 10 isolates the enriched cyclic Banach principle in the controlled setting; the regularity device is used consistently and no auxiliary smallness condition on the control function is required.

**Corollary 11.** Under the hypotheses of Theorem 10, suppose that for all  $x \in A_j$  and  $y \in A_{j+1}$ ,

$$d_\omega(T_\lambda x, T_\lambda y) \leq \theta d_\omega(x, y), \quad \theta \in [0, 1). \quad (14)$$

Then  $T_\lambda$  has a unique fixed point  $x^* \in \bigcap_{j=1}^p A_j$ , the iteration (6) converges to  $x^*$ , and

$$d_\omega(x_n, x^*) \leq M \frac{\theta^n}{1-\theta} d_\omega(x_0, x_1), \quad n \geq 0.$$



**Fig. 2.** Effect of the Krasnoselskij parameter  $\lambda$  on the decay of  $|x_n - x^*|$  for the averaged linear model with  $k = -0.8$

*Proof.* This is the case  $a = 0$  of Theorem 10, for which  $\varepsilon = \theta$ . □

We now exhibit an explicit instance of Theorem 10 in which the contractive inequality holds with  $\theta = 0$  and  $a = \frac{1}{3}$ , so that  $\varepsilon = \frac{1}{2}$ . The averaging identity, the cyclic inclusions, and the contractive inequality are checked below by direct computation.

**Example 12.** Let  $X = \mathbb{R}$  with the usual metric, which is a controlled metric with  $\omega \equiv 1$  and is continuous, and with the convex structure  $W(x, y, \lambda) = (1 - \lambda)x + \lambda y$ . Put:

$$A_1 = [0, 1], \quad A_2 = [-1, 0], \quad Y = A_1 \cup A_2,$$

so that  $A_1 \cap A_2 = \{0\}$ . Define  $T : Y \rightarrow \mathbb{R}$  by:

$$Tx = \begin{cases} -\frac{x^2 + 6x}{x + 4}, & x \in A_1, \\ -\frac{3x}{2}, & x \in A_2, \end{cases}$$

and take  $\lambda = \frac{1}{2}$ , so that  $T_\lambda x = \frac{1}{2}(x + Tx)$  becomes:

$$T_\lambda x = \begin{cases} -\frac{x}{x + 4}, & x \in A_1, \\ -\frac{x}{4}, & x \in A_2. \end{cases}$$

The two branches of  $T_\lambda$  arise from the averaging identity  $T_\lambda x = \frac{1}{2}(x + Tx)$ : for  $x \in A_1$ ,

$$T_\lambda x = \frac{1}{2} \left( x - \frac{x^2 + 6x}{x + 4} \right) = \frac{1}{2} \cdot \frac{x(x + 4) - (x^2 + 6x)}{x + 4} = \frac{1}{2} \cdot \frac{-2x}{x + 4} = -\frac{x}{x + 4},$$

and for  $x \in A_2$ ,  $T_\lambda x = \frac{1}{2} \left( x - \frac{3x}{2} \right) = -\frac{x}{4}$ . Since  $x \mapsto -x/(x + 4)$  is decreasing on  $[0, 1]$  from 0 to  $-\frac{1}{5}$ , one has  $T_\lambda(A_1) = [-\frac{1}{5}, 0] \subseteq A_2$ ; likewise  $T_\lambda(A_2) = [0, \frac{1}{4}] \subseteq A_1$ , so  $\{A_1, A_2\}$  is a cyclic representation

with respect to  $T_\lambda$ . We verify the enriched Kannan inequality with  $\theta = 0$  and  $a = \frac{1}{3}$ . Take  $x \in A_1$  and  $y \in A_2$ . The displacements are

$$d_\omega(x, T_\lambda x) = \left| x + \frac{x}{x+4} \right| = \frac{x(x+5)}{x+4}, \quad d_\omega(y, T_\lambda y) = \left| y + \frac{y}{4} \right| = \frac{5|y|}{4},$$

where the first uses  $x \geq 0$  and the second  $y \leq 0$ . Because  $x/(x+4) \in [0, \frac{1}{5}]$  and  $y/4 \in [-\frac{1}{4}, 0]$ , the image gap is

$$d_\omega(T_\lambda x, T_\lambda y) = \left| -\frac{x}{x+4} + \frac{y}{4} \right| = \frac{x}{x+4} + \frac{|y|}{4}.$$

Comparing this with  $\frac{1}{3}[d_\omega(x, T_\lambda x) + d_\omega(y, T_\lambda y)] = \frac{1}{3} \cdot \frac{x(x+5)}{x+4} + \frac{5|y|}{12}$  term by term, the coefficient of  $x/(x+4)$  obeys  $1 \leq \frac{1}{3}(x+5)$  for  $x \geq -2$ , and the coefficient of  $|y|$  obeys  $\frac{1}{4} \leq \frac{5}{12}$ ; hence

$$d_\omega(T_\lambda x, T_\lambda y) \leq \frac{1}{3}[d_\omega(x, T_\lambda x) + d_\omega(y, T_\lambda y)], \quad x \in A_1, y \in A_2, \quad (15)$$

and the same computation applies for  $x \in A_2, y \in A_1$ . The slack in Eq. (15), measured by the quotient  $d_\omega(T_\lambda x, T_\lambda y) / \frac{1}{3}[d_\omega(x, T_\lambda x) + d_\omega(y, T_\lambda y)]$ , attains its largest value  $0.6 < 1$  at the endpoint pair  $(x, y) = (0, -1)$  and its mirror image, confirming that Eq. (5) holds with  $\theta = 0, a = \frac{1}{3}$ , and therefore  $\varepsilon = (a + \theta)/(1 - a) = \frac{1}{2}$ . All hypotheses of Theorem 10 hold with  $M = 1$ , and the unique fixed point is  $x^* = 0 \in A_1 \cap A_2$ . Each step at least halves the displacement, since  $d_\omega(x_{n+1}, x_{n+2}) \leq \frac{1}{2}d_\omega(x_n, x_{n+1})$ , and the a priori estimate (8) gives  $d_\omega(x_n, x^*) \leq 2^{1-n}d_\omega(x_0, x_1)$ .

When a global contraction constant is not available but the space is compact, the strict contractive inequality still suffices. The following Edelstein type theorem in the controlled cyclic setting extends Edelstein [7].

**Theorem 13.** Let  $(X, d_\omega)$  be a controlled metric space with continuous  $d_\omega$ , let  $A_1, \dots, A_p$  be nonempty closed subsets with  $Y = \bigcup_{j=1}^p A_j$ , and let  $X$  carry a convex structure  $W$ . Let  $T : Y \rightarrow X$  and, for fixed  $\lambda \in (0, 1]$ , let  $T_\lambda$  be averaged as in (2) with  $\{A_j\}$  cyclic with respect to  $T_\lambda$ . Suppose that  $A = \bigcap_{j=1}^p A_j$  is nonempty and compact, that  $T_\lambda(A) \subseteq A$ , that  $T_\lambda|_A$  is continuous, and that for all  $x, y \in A$  with  $x \neq y$ ,

$$d_\omega(T_\lambda x, T_\lambda y) < d_\omega(x, y). \quad (16)$$

Then  $T_\lambda$  has a unique fixed point  $x^* \in A$ . If, in addition,  $\text{Fix}(T) = \text{Fix}(T_\lambda)$ , then  $T$  has a unique fixed point in  $Y$ .

*Proof.* Since  $A$  is nonempty and compact and  $T_\lambda(A) \subseteq A$ , the restriction  $T_\lambda|_A$  is a continuous self-map of the compact set  $A$  that obeys the strict contractive inequality (16). Consider the displacement functional

$$\varphi : A \rightarrow [0, \infty), \quad \varphi(x) = d_\omega(x, T_\lambda x).$$

The map  $x \mapsto (x, T_\lambda x)$  is continuous from  $A$  into  $A \times A$  because  $T_\lambda|_A$  is continuous, and  $d_\omega$  is continuous by hypothesis; hence  $\varphi$  is continuous on  $A$ . A continuous real-valued function on a nonempty compact set attains its infimum, so there exists  $x^* \in A$  with  $\varphi(x^*) = \min_{x \in A} \varphi(x)$ . Suppose, toward a contradiction, that  $T_\lambda x^* \neq x^*$ . Both  $x^*$  and  $T_\lambda x^*$  lie in  $A$ , and  $T_\lambda^2 x^* \in A$  as well because  $T_\lambda(A) \subseteq A$ ; therefore, Eq. (16) applies to the pair  $(x^*, T_\lambda x^*)$  and gives

$$\varphi(T_\lambda x^*) = d_\omega(T_\lambda x^*, T_\lambda(T_\lambda x^*)) = d_\omega(T_\lambda x^*, T_\lambda^2 x^*) < d_\omega(x^*, T_\lambda x^*) = \varphi(x^*),$$

which contradicts the minimality of  $\varphi(x^*)$ . Hence  $T_\lambda x^* = x^*$ . For uniqueness, let  $u^* = T_\lambda u^*$  with  $u^* \neq x^*$ ; then  $u^* \in A$  and Eq. (16) yields

$$d_\omega(x^*, u^*) = d_\omega(T_\lambda x^*, T_\lambda u^*) < d_\omega(x^*, u^*),$$

which is impossible. Thus  $x^* = u^*$ , and the statement for  $T$  follows from  $\text{Fix}(T) = \text{Fix}(T_\lambda)$ .  $\square$

A further specialization of the generalized Reich condition is the enriched cyclic Kannan type contraction, in which the contractive bound involves only the displacement at the first argument. We record it with the sharp threshold on the constant; the proof makes explicit that the displacement term does not vanish but instead contributes to the effective contraction factor.

**Theorem 14.** *Let the hypotheses on  $(X, d_\omega)$ ,  $W$ ,  $\{A_j\}$ ,  $T$ ,  $T_\lambda$ , and the regularity device (7) be as in Theorem 10, with  $d_\omega$  continuous. Suppose there exists  $\xi \in [0, \frac{1}{3})$  such that for all  $x \in A_j$  and  $y \in A_{j+1}$ ,*

$$d_\omega(T_\lambda x, T_\lambda y) \leq \xi d_\omega(x, y) + 2\xi d_\omega(x, T_\lambda x). \quad (17)$$

*Then, the iteration (6) converges to the unique fixed point  $x^* \in \bigcap_{j=1}^p A_j$  of  $T_\lambda$ , and*

$$d_\omega(x_n, x^*) \leq M \frac{(3\xi)^n}{1 - 3\xi} d_\omega(x_0, x_1), \quad d_\omega(x_{n+1}, x^*) \leq M \frac{3\xi}{1 - 3\xi} d_\omega(x_n, x_{n+1}). \quad (18)$$

*If  $\text{Fix}(T) = \text{Fix}(T_\lambda)$ , then  $T$  has a unique fixed point in  $Y$ .*

*Proof.* Apply Eq. (17) with  $x = x_n \in A_{j+n}$  and  $y = x_{n+1} \in A_{j+n+1}$ . Since  $T_\lambda x_n = x_{n+1}$ , the displacement at the first argument is  $d_\omega(x_n, T_\lambda x_n) = d_\omega(x_n, x_{n+1})$ , and therefore:

$$d_\omega(x_{n+1}, x_{n+2}) \leq \xi d_\omega(x_n, x_{n+1}) + 2\xi d_\omega(x_n, x_{n+1}) = 3\xi d_\omega(x_n, x_{n+1}).$$

The hypothesis  $\xi \in [0, \frac{1}{3})$  is exactly  $3\xi \in [0, 1)$ , so the displacement contracts by the factor  $3\xi$  at every step. An induction identical to Eq. (12) gives  $d_\omega(x_n, x_{n+1}) \leq (3\xi)^n d_\omega(x_0, x_1)$ , and the regularity device (7) together with the finite geometric sum  $\sum_{k=n}^{m-1} (3\xi)^k = (3\xi)^n \frac{1 - (3\xi)^{m-n}}{1 - 3\xi}$  yields:

$$\begin{aligned} d_\omega(x_n, x_m) &\leq M \sum_{k=n}^{m-1} (3\xi)^k d_\omega(x_0, x_1) = M \frac{(3\xi)^n (1 - (3\xi)^{m-n})}{1 - 3\xi} d_\omega(x_0, x_1) \\ &\leq M \frac{(3\xi)^n}{1 - 3\xi} d_\omega(x_0, x_1) \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

so  $\{x_n\}$  is Cauchy and converges to some  $x^* \in Y$ ; letting  $m \rightarrow \infty$  yields the first estimate in Eq. (18), and the second follows as in Theorem 10. As before,  $x^* \in \bigcap_j A_j$  because each  $A_j$  is closed. For the fixed point property, continuity of  $d_\omega$  gives

$$d_\omega(x_{n+1}, T_\lambda x^*) = d_\omega(T_\lambda x_n, T_\lambda x^*) \leq \xi d_\omega(x_n, x^*) + 2\xi d_\omega(x_n, x_{n+1}) \xrightarrow{n \rightarrow \infty} 0,$$

while  $d_\omega(x_{n+1}, T_\lambda x^*) \rightarrow d_\omega(x^*, T_\lambda x^*)$ . Hence,  $d_\omega(x^*, T_\lambda x^*) = 0$  and  $T_\lambda x^* = x^*$ . No additional condition on  $\omega(x^*, T_\lambda x^*)$  is needed. Uniqueness follows from Eq. (17): if  $u^* = T_\lambda u^*$  then  $d_\omega(x^*, u^*) \leq \xi d_\omega(x^*, u^*)$ , and  $\xi < 1$  forces  $x^* = u^*$ . The statement for  $T$  is immediate when  $\text{Fix}(T) = \text{Fix}(T_\lambda)$ .  $\square$

The three convergence theorems of this section are governed by a single mechanism, in which averaging replaces the underlying operator by  $T_\lambda$  and the effective one-step factor is read off from the enriched Reich constants. Table 1 collects the contractive conditions, the effective factors, and the a priori estimates side by side, and Figure 3 visualizes the corresponding surfaces in panels (d)–(f) and (o). In Figure 3, panels (a)–(c) show the admissible control functions  $\omega_1(x, y) = 1 + |x| + |y|$ ,  $\omega_2(x, y) = 1 + x^2 + y^2$ , and  $\omega_3(x, y) = 1 + |\sin x| + |\sin y|$  of Definition 1. Panel (d) is the effective convergence factor  $\varepsilon = (a + \theta)/(1 - a)$  over the admissible region  $\theta < 1 - 2a$ , and panel (e) is

the Krasnoselskij one-step factor  $|1 - \lambda + \lambda k|$  for the averaged linear model. Panel (f) is the base-ten logarithm of the a priori bound  $\varepsilon^n / (1 - \varepsilon)$  of Theorem 10. Panel (g) is the Bielecki weight  $e^{-\tau t}$  underlying the metric  $d_\tau$ , panel (h) shows the Picard iterates  $\mathcal{P}^k u_0$  approaching  $e^{-t}$  (solid curve), panel (i) is the network output  $u_\vartheta$  across training epochs, and panel (j) is the base-ten logarithm of the training loss on a two-dimensional slice through the trained parameters. Panels (k)–(n) are attractors of three-dimensional iterated function systems, namely a Sierpiński tetrahedron, a Cantor dust, a self-similar tree, and a Menger sponge. Panel (o) is the base-ten logarithm of  $|x_n - x^*|$  for the averaged linear model as a function of the iteration  $n$  and the averaging parameter  $\lambda$ .

**Table 1**

Convergence results for averaged mappings on convex controlled metric spaces, with the a priori error measured against the first step  $d_\omega(x_0, x_1)$

| Result                                   | Contractive condition on $T_\lambda$                                                                                     | Effective factor                         | A priori error $d_\omega(x_n, x^*) \leq$                     |
|------------------------------------------|--------------------------------------------------------------------------------------------------------------------------|------------------------------------------|--------------------------------------------------------------|
| Theorem 10<br>(enriched cyclic Reich)    | $d_\omega(T_\lambda x, T_\lambda y) \leq \theta d_\omega(x, y) + a[d_\omega(x, T_\lambda x) + d_\omega(y, T_\lambda y)]$ | $\varepsilon = \frac{a + \theta}{1 - a}$ | $M \frac{\varepsilon^n}{1 - \varepsilon} d_\omega(x_0, x_1)$ |
| Corollary 11<br>(enriched cyclic Banach) | $d_\omega(T_\lambda x, T_\lambda y) \leq \theta d_\omega(x, y)$                                                          | $\theta$                                 | $M \frac{\theta^n}{1 - \theta} d_\omega(x_0, x_1)$           |
| Theorem 14<br>(enriched cyclic Kannan)   | $d_\omega(T_\lambda x, T_\lambda y) \leq \xi d_\omega(x, y) + 2\xi d_\omega(x, T_\lambda x),$<br>$\xi < \frac{1}{3}$     | $3\xi$                                   | $M \frac{(3\xi)^n}{1 - 3\xi} d_\omega(x_0, x_1)$             |
| Theorem 13<br>(Edelstein, compact)       | $d_\omega(T_\lambda x, T_\lambda y) < d_\omega(x, y)$<br>on compact $A$                                                  | strict, no constant                      | existence and uniqueness on $A$                              |

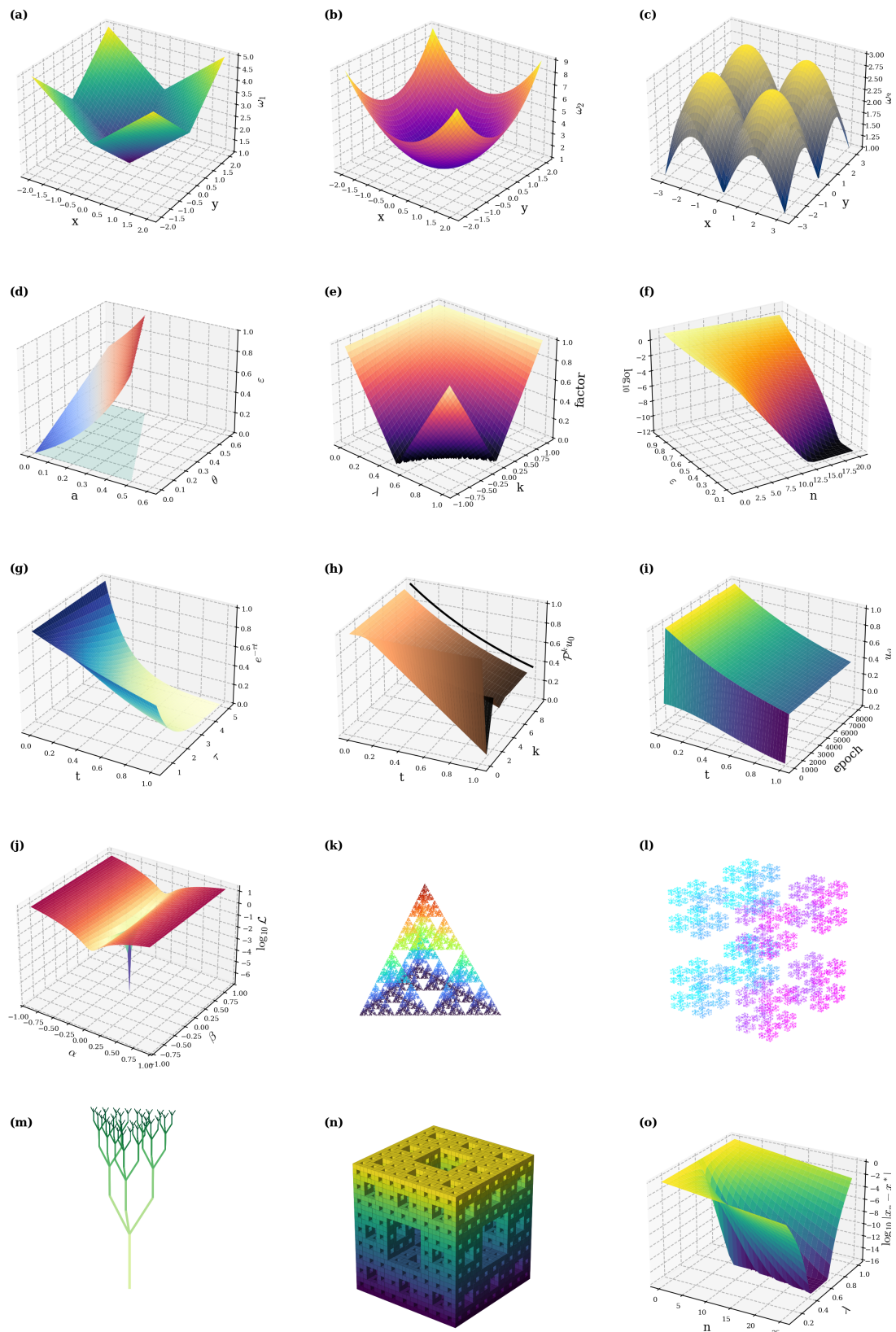
## 4. Application to Generalized Iterated Function Systems

We now use the abstract theory to construct fractal attractors. Let  $(X, d_\omega)$  be a complete controlled metric space with continuous  $d_\omega$  and a convex structure  $W$ , and recall the controlled Pompeiu–Hausdorff metric of Definition 7. The next theorem shows that the  $H_\omega$ -contractivity of the induced set map is inherited from a cyclic contraction of the underlying point map. To keep the geometry of the hyperspace transparent, we work with the contractive form in which the averaged map satisfies a single-constant estimate; this is the case  $a = 0, p = 1$  of Eq. (5), in which the cyclic condition is vacuous and  $T_\lambda$  is a self-map of  $X$ , exactly as in the classical Hutchinson–Barnsley theory.

**Remark 15.** Taking  $p = 1, A_1 = X, \lambda = 1$ , and  $a = 0$  in Definition 6 reduces a generalized enriched cyclic contraction to a classical Banach contraction  $d_\omega(Tx, Ty) \leq \theta d_\omega(x, y)$  on the whole space. Thus the framework of Section 3 contains the Hutchinson [30] and Barnsley [31] setting as a special case, and the attractor theory below specializes to the classical one.

**Theorem 16.** Let  $T : X \rightarrow X$  map nonempty compact sets to nonempty compact sets, and suppose that for some  $\theta \in [0, 1)$  and  $\lambda \in (0, 1]$  the averaged map satisfies

$$d_\omega(T_\lambda x, T_\lambda y) \leq \theta \lambda d_\omega(x, y), \quad x, y \in X. \quad (19)$$



**Fig. 3.** Surfaces and attractors generated from the models

Then the induced set map  $(T_\lambda)^* : C(X) \rightarrow C(X)$  satisfies

$$H_\omega((T_\lambda)^* A, (T_\lambda)^* B) \leq \theta \lambda H_\omega(A, B), \quad A, B \in C(X). \quad (20)$$

*Proof.* Fix  $A, B \in C(X)$  and  $u \in (T_\lambda)^* A = T_\lambda(A)$ , so  $u = T_\lambda a$  for some  $a \in A$ . For every  $b \in B$ ,

$$d_\omega(u, (T_\lambda)^* B) = \inf_{v \in T_\lambda(B)} d_\omega(u, v) \leq d_\omega(T_\lambda a, T_\lambda b) \leq \theta \lambda d_\omega(a, b).$$

Taking the infimum over  $b \in B$  gives  $d_\omega(u, (T_\lambda)^* B) \leq \theta \lambda d_\omega(a, B) \leq \theta \lambda D_\omega(A, B)$ , and the supremum over  $u$  yields  $D_\omega((T_\lambda)^* A, (T_\lambda)^* B) \leq \theta \lambda D_\omega(A, B)$ . The symmetric estimate holds by the same argument, and taking the maximum gives Eq. (20).  $\square$

**Corollary 17.** Let  $\mathcal{T} = \{T^{(1)}, \dots, T^{(N)}\}$  be a finite family of maps as in Theorem 16, each satisfying Eq.(19) with constants  $\theta_n \lambda_n$ , and let  $F$  be the associated Hutchinson operator (Eq. (4)). If  $\theta^* = \max_{1 \leq n \leq N} \theta_n \lambda_n < 1$ , then  $F$  is a contraction on the complete metric space  $(C(X), H_\omega)$  with constant  $\theta^*$ , and consequently  $F$  possesses a unique attractor  $A^* \in C(X)$ , that is,  $F(A^*) = A^*$ . Moreover, for every initial  $B_0 \in C(X)$  the orbit  $B_{k+1} = F(B_k)$  converges to  $A^*$  with

$$H_\omega(B_k, A^*) \leq (\theta^*)^k H_\omega(B_0, A^*), \quad k \geq 0.$$

*Proof.* By Lemma 8 and Theorem 16:

$$H_\omega(F(A), F(B)) = H_\omega\left(\bigcup_n (T_\lambda^{(n)})^* A, \bigcup_n (T_\lambda^{(n)})^* B\right) \leq \max_n H_\omega((T_\lambda^{(n)})^* A, (T_\lambda^{(n)})^* B) \leq \theta^* H_\omega(A, B).$$

Since  $(X, d_\omega)$  is complete, the hyperspace  $(C(X), H_\omega)$  is complete, and the Banach principle applied to the contraction  $F$  yields a unique fixed point  $A^*$  together with the stated geometric convergence. For the orbit bound, iterate the contraction: writing  $B_k = F^k(B_0)$  and using  $F(A^*) = A^*$ ,

$$H_\omega(B_k, A^*) = H_\omega(F(B_{k-1}), F(A^*)) \leq \theta^* H_\omega(B_{k-1}, A^*) \leq \dots \leq (\theta^*)^k H_\omega(B_0, A^*),$$

which is the stated estimate.  $\square$

A practical companion to the convergence estimate is the collage bound, which measures the distance from an arbitrary set to the attractor by how nearly that set is left invariant by  $F$ .

**Corollary 18** (Collage bound). In the setting of Corollary 17 with  $\omega \equiv 1$ , so that  $H_\omega$  is the Pompeiu-Hausdorff metric, every  $B \in C(X)$  satisfies:

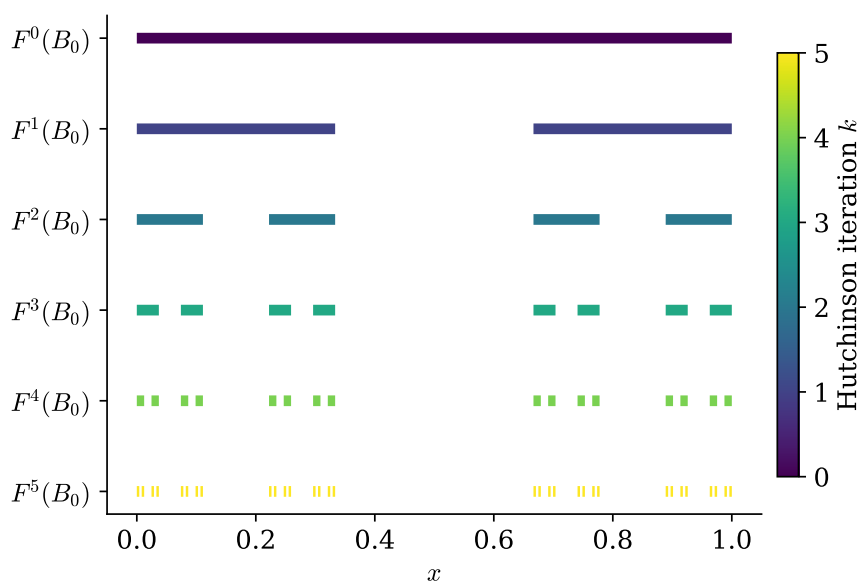
$$H_\omega(B, A^*) \leq \frac{1}{1 - \theta^*} H_\omega(B, F(B)). \quad (21)$$

*Proof.* Because  $\omega \equiv 1$ , the metric  $H_\omega$  obeys the triangle inequality. Inserting  $F(B)$  and using  $A^* = F(A^*)$  with the contraction of Corollary 17,

$$H_\omega(B, A^*) \leq H_\omega(B, F(B)) + H_\omega(F(B), F(A^*)) \leq H_\omega(B, F(B)) + \theta^* H_\omega(B, A^*).$$

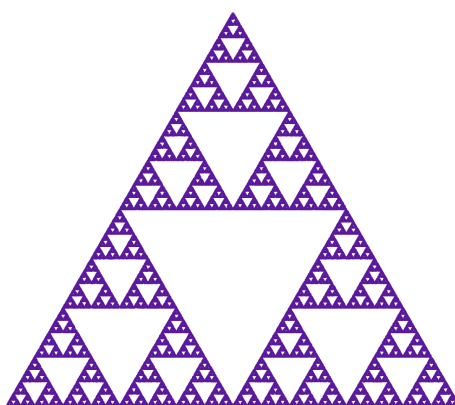
Since  $\theta^* < 1$ , subtracting  $\theta^* H_\omega(B, A^*)$  and dividing by  $1 - \theta^* > 0$  gives Eq. (21).  $\square$

The classical fractals arise as attractors of such systems and provide concrete illustrations. Figure 4 shows the deterministic Hutchinson orbit  $B_{k+1} = F(B_k)$  of the two-map system  $f_1(x) = x/3$ ,  $f_2(x) = x/3 + 2/3$  on  $[0, 1]$ , whose attractor is the Cantor set; each map is a similarity with ratio  $1/3$ , so  $\theta^* = 1/3$ .



**Fig. 4.** Deterministic Hutchinson orbit of the two-similarity system  $f_1(x) = x/3$ ,  $f_2(x) = x/3 + 2/3$ , converging to the Cantor attractor

Figure 5 displays the attractor of the three-map system of plane similarities  $x \mapsto \frac{1}{2}(x + v_i)$  with vertices  $v_1, v_2, v_3$ , namely the Sierpiński gasket, for which  $\theta^* = 1/2$ . Figure 6 exhibits the Barnsley fern generated by a four-map affine system, demonstrating the expressive range of the construction beyond strictly self-similar sets. The same collage mechanism operates in three dimensions: the Sierpiński tetrahedron, the Cantor dust, the self-similar tree, and the Menger sponge of Figure 3(k)–(n) are attractors of spatial iterated function systems whose maps are contractions of ratio  $1/2$  or  $1/3$ , so that  $\theta^* < 1$  and Corollary 17 applies verbatim.

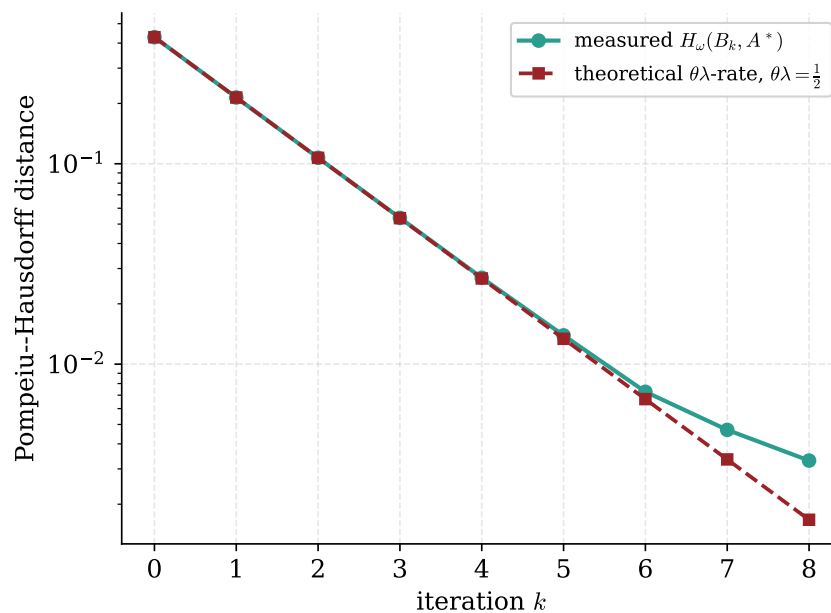


**Fig. 5.** Sierpiński gasket, the attractor of three plane similarities of ratio  $1/2$ , for which  $\theta^* = 1/2$

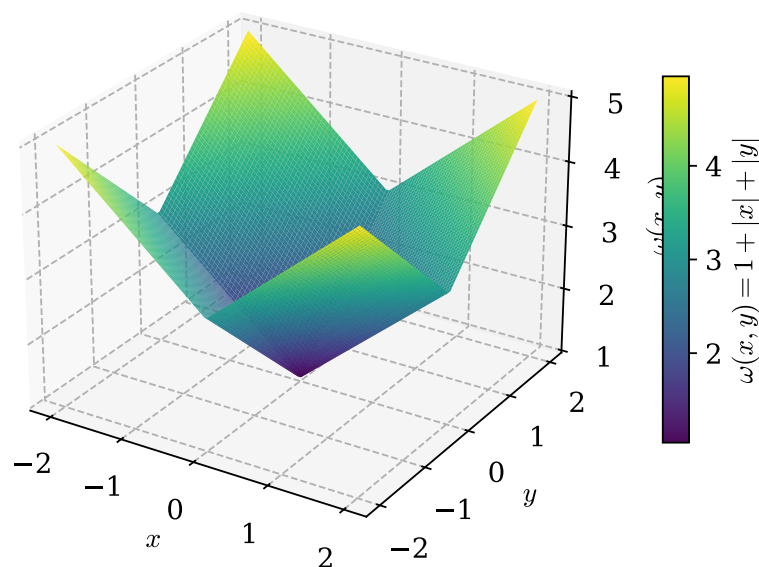


**Fig. 6.** Barnsley fern produced by a four-map affine iterated function system, illustrating the generality of the framework

The geometric convergence guaranteed by Corollary 17 is verified numerically in Figure 7, where the measured  $H_\omega$ -distance between the Hutchinson orbit and the attractor of the Sierpiński system decays at the predicted rate  $\theta\lambda = \frac{1}{2}$ . Finally, Figure 8 renders a control function  $\omega(x, y) = 1 + |x| + |y|$ , illustrating the point-dependent moderation of the triangle inequality that distinguishes the controlled setting from the metric and  $b$ -metric cases; the related admissible surfaces  $\omega_2$  and  $\omega_3$  appear in Figure 3(a)–(c).



**Fig. 7.** Measured controlled Pompeiu–Hausdorff distance between the Hutchinson orbit and the attractor against the predicted geometric rate  $\theta\lambda = \frac{1}{2}$



**Fig. 8.** Control function  $\omega(x, y) = 1 + |x| + |y|$  on the plane, which moderates the controlled triangle inequality in a point-dependent manner

## 5. Physics-Informed Neural Network Realization

We close the loop between the abstract theory and computation by realizing the fixed point of a concrete operator through a physics-informed neural network in the sense of Raissi, Perdikaris, and Karniadakis [36]. Consider the initial value problem:

$$u'(t) + u(t) = 0, \quad t \in [0, 1], \quad u(0) = 1, \quad (22)$$

whose exact solution is  $u^*(t) = e^{-t}$ . To recast Eq. (22) in integral form, integrate the identity  $u'(s) = -u(s)$  over  $[0, t]$  and apply the fundamental theorem of calculus together with  $u(0) = 1$ :

$$u(t) - 1 = u(t) - u(0) = \int_0^t u'(s) ds = - \int_0^t u(s) ds,$$

so that  $u(t) = 1 - \int_0^t u(s) ds$ . Conversely, if a continuous  $u$  satisfies this relation then the right-hand side is differentiable with derivative  $-u(t)$ , whence  $u'(t) + u(t) = 0$ , and evaluating at  $t = 0$  gives  $u(0) = 1$ . Therefore  $u \in C[0, 1]$  solves Eq. (22) if and only if it is a fixed point of the Picard operator:

$$(\mathcal{P}u)(t) = 1 - \int_0^t u(s) ds. \quad (23)$$

Equip  $C[0, 1]$  with the Bielecki weighted metric of Bielecki [37]:

$$d_\tau(u, v) = \sup_{t \in [0, 1]} e^{-\tau t} |u(t) - v(t)|, \quad \tau > 0,$$

which is a metric on  $C[0, 1]$  and therefore a controlled metric with  $\omega \equiv 1$ , and which is continuous. Since  $\mathcal{P}$  is affine, the difference of two images telescopes to a single integral,  $(\mathcal{P}u)(t) - (\mathcal{P}v)(t) = \int_0^t (v(s) - u(s)) ds$ . Writing  $|v(s) - u(s)| = e^{\tau s} (e^{-\tau s} |u(s) - v(s)|) \leq e^{\tau s} d_\tau(u, v)$  and evaluating the elementary integral  $\int_0^t e^{\tau s} ds = \frac{e^{\tau t} - 1}{\tau}$ , we obtain for every  $t \in [0, 1]$ :

$$\begin{aligned} e^{-\tau t} |(\mathcal{P}u)(t) - (\mathcal{P}v)(t)| &= e^{-\tau t} \left| \int_0^t (v(s) - u(s)) ds \right| \leq e^{-\tau t} \int_0^t |v(s) - u(s)| ds \\ &\leq e^{-\tau t} d_\tau(u, v) \int_0^t e^{\tau s} ds = e^{-\tau t} \frac{e^{\tau t} - 1}{\tau} d_\tau(u, v) \\ &= \frac{1 - e^{-\tau t}}{\tau} d_\tau(u, v) \leq \frac{1}{\tau} d_\tau(u, v), \end{aligned} \quad (24)$$

where the last inequality uses  $1 - e^{-\tau t} \leq 1$ . Taking the supremum over  $t \in [0, 1]$  on the left gives

$$d_\tau(\mathcal{P}u, \mathcal{P}v) \leq \frac{1}{\tau} d_\tau(u, v). \quad (25)$$

For any  $\tau > 1$  the operator  $\mathcal{P}$  is a contraction with constant  $q := 1/\tau < 1$ , hence by Corollary 11 with  $p = 1$  and  $a = 0$  it has a unique fixed point, namely  $u^* = e^{-t}$ , and the Picard iterates  $\mathcal{P}^k u_0$  converge to it geometrically. The averaged operator  $\mathcal{P}_\lambda = (1 - \lambda)I + \lambda\mathcal{P}$  is an enriched contraction with the same fixed point, in exact correspondence with the averaging device of Definition 5.

The contraction constant translates into an explicit, uniform error bound for the iterates.

**Proposition 19.** *Let  $\tau > 1$ ,  $q = 1/\tau$ , and let  $u_0 \in C[0, 1]$ . Then, for every  $k \geq 0$*

$$d_\tau(\mathcal{P}^k u_0, u^*) \leq \frac{q^k}{1 - q} d_\tau(u_0, \mathcal{P}u_0), \quad (26)$$

and, since  $e^{-\tau} \|f\|_\infty \leq d_\tau(f, 0) \leq \|f\|_\infty$  on  $[0, 1]$ , the uniform error obeys:

$$\|\mathcal{P}^k u_0 - u^*\|_\infty \leq e^\tau \frac{q^k}{1 - q} d_\tau(u_0, \mathcal{P}u_0) \xrightarrow[k \rightarrow \infty]{} 0. \quad (27)$$

**Proof.** From Eq. (25) the iterates satisfy  $d_\tau(\mathcal{P}^{k+1}u_0, \mathcal{P}^k u_0) \leq q d_\tau(\mathcal{P}^k u_0, \mathcal{P}^{k-1} u_0) \leq q^k d_\tau(u_0, \mathcal{P}u_0)$ . For  $m > k$  the triangle inequality of the metric  $d_\tau$  and the geometric series give:

$$d_\tau(\mathcal{P}^k u_0, \mathcal{P}^m u_0) \leq \sum_{j=k}^{m-1} d_\tau(\mathcal{P}^j u_0, \mathcal{P}^{j+1} u_0) \leq d_\tau(u_0, \mathcal{P}u_0) \sum_{j=k}^{m-1} q^j \leq \frac{q^k}{1-q} d_\tau(u_0, \mathcal{P}u_0).$$

Letting  $m \rightarrow \infty$  and using  $\mathcal{P}^m u_0 \rightarrow u^*$  yields Eq. (26). The norm equivalence bounds follow from  $e^{-\tau t} \in [e^{-\tau}, 1]$  for  $t \in [0, 1]$ , which gives Eq. (27).  $\square$

A physics-informed neural network approximates this fixed point directly. We represent the candidate solution by a single-hidden-layer network with  $H = 64$  hyperbolic-tangent units:

$$u_\vartheta(t) = \sum_{j=1}^H v_j \sigma_j(t) + c, \quad \sigma_j(t) = \tanh(w_j t + b_j), \quad \vartheta = (w, b, v, c). \quad (28)$$

Writing  $\sigma'_j(t) = w_j \operatorname{sech}^2(w_j t + b_j) = w_j(1 - \sigma_j^2(t))$ , the time derivative and the equation residual are available in closed form:

$$u'_\vartheta(t) = \sum_{j=1}^H v_j w_j (1 - \sigma_j^2(t)), \quad r_\vartheta(t) := u'_\vartheta(t) + u_\vartheta(t) = \sum_{j=1}^H v_j [w_j (1 - \sigma_j^2(t)) + \sigma_j(t)] + c. \quad (29)$$

The parameters are trained from scratch to minimize the composite loss:

$$\mathcal{L}(\vartheta) = \frac{1}{N} \sum_{i=1}^N r_\vartheta(t_i)^2 + \beta (u_\vartheta(0) - 1)^2, \quad (30)$$

with  $N = 200$  collocation points  $t_i$  uniformly spaced on  $[0, 1]$ , penalty weight  $\beta = 20$ , and no automatic-differentiation library. Abbreviating  $s_j = \sigma_j(t)$  and  $\operatorname{sech}_j^2 = 1 - s_j^2$ , the residual sensitivities are

$$\frac{\partial r_\vartheta}{\partial v_j} = w_j \operatorname{sech}_j^2 + s_j, \quad \frac{\partial r_\vartheta}{\partial w_j} = v_j \operatorname{sech}_j^2 (1 + t - 2w_j s_j t), \quad \frac{\partial r_\vartheta}{\partial b_j} = v_j \operatorname{sech}_j^2 (1 - 2w_j s_j), \quad \frac{\partial r_\vartheta}{\partial c} = 1, \quad (31)$$

and, with  $\iota = u_\vartheta(0) - 1$  and  $s_{0j} = \tanh(b_j)$ , the gradient of Eq. (30) reads:

$$\begin{aligned} \nabla_\vartheta \mathcal{L} &= \frac{2}{N} \sum_{i=1}^N r_\vartheta(t_i) \nabla_\vartheta r_\vartheta(t_i) + 2\beta \iota \nabla_\vartheta u_\vartheta(0), \\ \frac{\partial u_\vartheta(0)}{\partial v_j} &= s_{0j}, \quad \frac{\partial u_\vartheta(0)}{\partial b_j} = v_j (1 - s_{0j}^2), \quad \frac{\partial u_\vartheta(0)}{\partial c} = 1, \end{aligned} \quad (32)$$

while  $\partial u_\vartheta(0)/\partial w_j = 0$  because the input vanishes at  $t = 0$ . Each parameter is updated by the Adam rule: with gradient  $g_t$  at step  $t$ :

$$m_t = \beta_1 m_{t-1} + (1 - \beta_1) g_t, \quad v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2, \quad \vartheta_t = \vartheta_{t-1} - \eta \frac{m_t / (1 - \beta_1^t)}{\sqrt{v_t / (1 - \beta_2^t) + \epsilon}}, \quad (33)$$

run for 8000 steps with  $\eta = 5 \times 10^{-3}$ ,  $\beta_1 = 0.9$ ,  $\beta_2 = 0.999$ , and  $\epsilon = 10^{-8}$ . After training, the surrogate attains a maximum pointwise error  $\|u_\vartheta - u^*\|_\infty = 4.586 \times 10^{-5}$ , a root-mean-square error  $(\frac{1}{M} \sum (u_\vartheta - u^*)^2)^{1/2} = 1.799 \times 10^{-5}$  over  $M = 401$  evaluation points (relative  $L^2$  error  $2.734 \times$

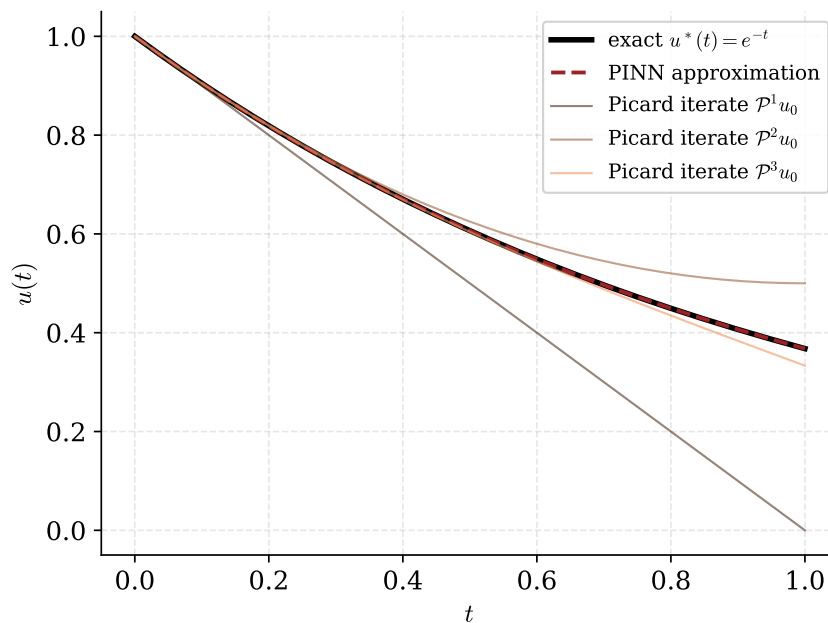
**Table 2**

Accuracy of the trained network for  $u' + u = 0$ ,  $u(0) = 1$  on  $[0, 1]$ , evaluated at  $M = 401$  points against  $u^*(t) = e^{-t}$

| Quantity                                  | Symbol                                                      | Measured value         |
|-------------------------------------------|-------------------------------------------------------------|------------------------|
| Maximum pointwise error                   | $\ u_\vartheta - u^*\ _\infty$                              | $4.586 \times 10^{-5}$ |
| Root-mean-square error                    | $\left(\frac{1}{M} \sum (u_\vartheta - u^*)^2\right)^{1/2}$ | $1.799 \times 10^{-5}$ |
| Relative $L^2$ error                      | $\ u_\vartheta - u^*\ _2 / \ u^*\ _2$                       | $2.734 \times 10^{-5}$ |
| Terminal training loss                    | $\mathcal{L}(\vartheta)$                                    | $1.754 \times 10^{-7}$ |
| Predicted iteration factor ( $\tau = 2$ ) | $q = 1/\tau$                                                | 0.5                    |

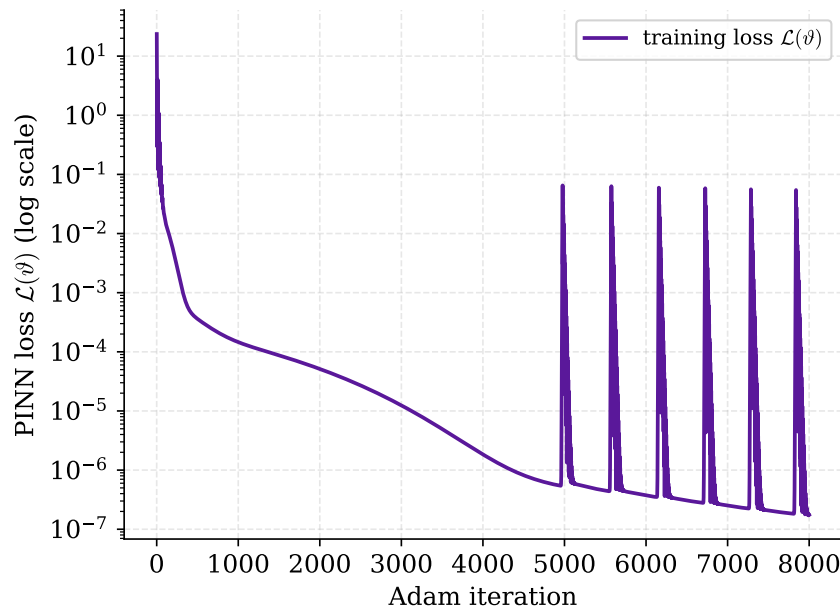
$10^{-5}$ ), and a terminal loss  $\mathcal{L}(\vartheta) = 1.754 \times 10^{-7}$ . These values are consistent with the geometric convergence predicted by the contraction estimate Eq. (25) and Proposition 19: the trained network realizes, to within the optimization tolerance, the unique fixed point whose existence and uniqueness are guaranteed by the abstract theory. The accuracy of the trained surrogate is summarized in Table 2.

Figure 9 overlays the network output, the exact solution, and the first Picard iterates  $\mathcal{P}^k u_0$ , exhibiting the convergence of the fixed point iteration alongside the neural approximation.

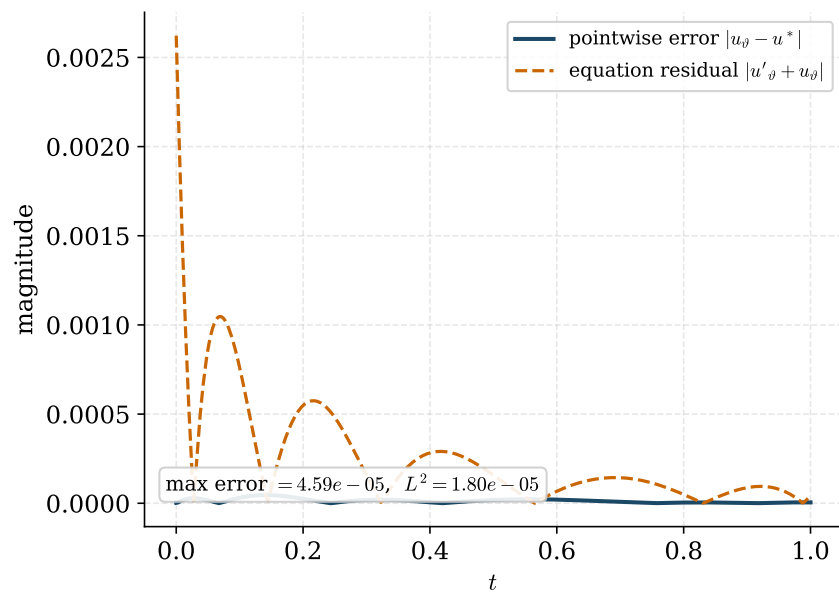


**Fig. 9.** Physics-informed neural network solution of Eq. (22) against the exact solution  $u^*(t) = e^{-t}$  and the Picard iterates  $\mathcal{P}^k u_0$  of the operator (Eq. (23))

Figure 10 shows the decay of the training loss (30) on a logarithmic scale, and Figure 11 reports the pointwise error and the equation residual across the interval, both of order  $10^{-5}$  or smaller. The Bielecki weight  $e^{-\tau t}$  that makes  $\mathcal{P}$  a contraction, the Picard iterate surface converging to  $e^{-t}$ , the network output across training epochs, and the loss landscape near the trained parameters are collected in Figure 3(g)–(j). The agreement between the measured neural error and the contraction-based prediction substantiates the claim that fixed point theory in controlled metric spaces provides a rigorous foundation for this class of scientific machine learning methods.



**Fig. 10.** Decay of the composite residual-plus-initial-data loss (30) during training, on a logarithmic scale



**Fig. 11.** Pointwise error  $|u_\vartheta - u^*|$  and equation residual  $|u'_\vartheta + u_\vartheta|$  of the trained network across the interval

## 6. Comparison with Existing Approaches

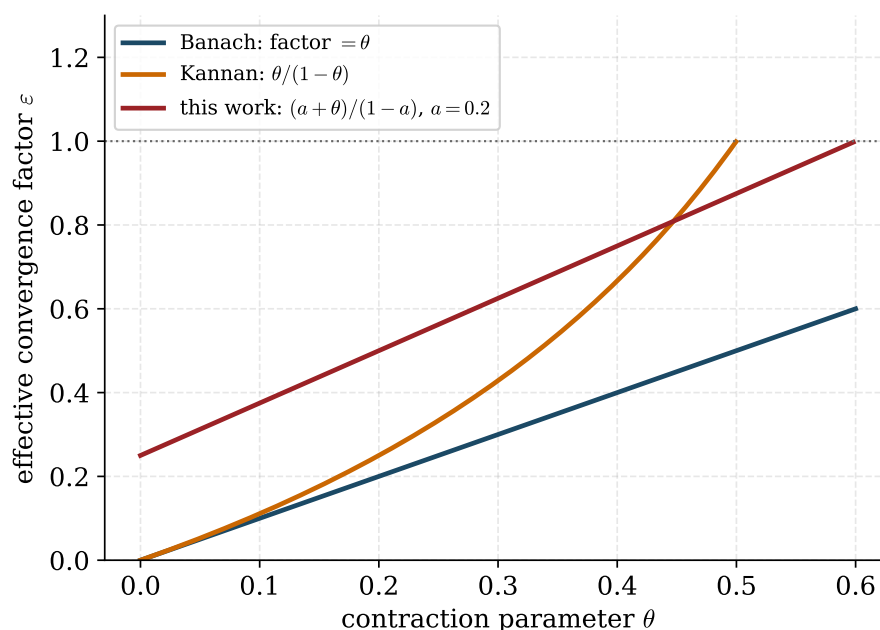
The framework developed above subsumes and strictly enlarges a number of established results. Table 3 summarizes the relationship between the present theory and several representative contraction principles, recording for each the ambient structure, the contractive condition, and the parameter regime under which it is recovered from Theorem 10 or its corollaries. The Banach principle [1] corresponds to  $p = 1$ ,  $\lambda = 1$ ,  $a = 0$  and  $\omega \equiv 1$ ; the Kannan principle [2] to  $\theta = 0$  in the symmetric form;

the Reich and Hardy–Rogers principles [4, 5] to the generalized linear condition with  $\omega \equiv 1$ ; the cyclic principle of Kirk, Srinivasan, and Veeramani [8] to  $a = 0$ ,  $\lambda = 1$  with  $p \geq 2$ ; the enriched contractions of Berinde and Păcurar [16, 17] to  $p = 1$  and  $\lambda \in (0, 1]$ ; the controlled metric type spaces of Mlaiki, Aydi, Souayah, and Abdeljawad [23] to a nontrivial control function with  $a = 0$ ,  $\lambda = 1$ ,  $p = 1$ ; the generalized enriched cyclic contractions of Abbas, Anjum, and Iqbal [20] to the metric case  $\omega \equiv 1$ ; and the Hutchinson–Barnsley attractor theory [30, 31] to the iterated-function-system specialization of Corollary 17.

**Table 3**  
Position of the present framework relative to existing contraction principles

| Approach                      | Ambient structure |                                | Contractive condition                         | Recovered when                                    |
|-------------------------------|-------------------|--------------------------------|-----------------------------------------------|---------------------------------------------------|
| Banach [1]                    | metric space      |                                | $d(Tx, Ty) \leq \theta d(x, y)$               | $\omega \equiv 1, a = 0, \lambda = 1, p = 1$      |
| Kannan [2]                    | metric space      |                                | $d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$       | $\omega \equiv 1, \theta = 0, \lambda = 1, p = 1$ |
| Reich, Hardy–Rogers [4, 5]    | metric space      |                                | generalized linear combination                | $\omega \equiv 1, \lambda = 1, p = 1$             |
| Edelstein [7]                 | compact space     | metric                         | $d(Tx, Ty) < d(x, y)$                         | Theorem 13, $\omega \equiv 1$                     |
| Kirk–Srinivasan–Veeramani [8] | metric space      |                                | cyclic contraction                            | $\omega \equiv 1, a = 0, \lambda = 1$             |
| Berinde–Păcurar [16, 17]      | normed space      |                                | enriched contraction                          | $\omega \equiv 1, p = 1$                          |
| Mlaiki et al. [23]            | controlled space  | metric                         | $d_\omega(Tx, Ty) \leq \theta d_\omega(x, y)$ | $a = 0, \lambda = 1, p = 1$                       |
| Abbas–Anjum–Iqbal [20]        | metric space      |                                | enriched cyclic contraction                   | $\omega \equiv 1$                                 |
| Hutchinson, Barnsley [30, 31] | complete space    | metric                         | contractive IFS                               | Corollary 17, $\omega \equiv 1$                   |
| <b>This work</b>              | <b>convex</b>     | <b>controlled metric space</b> | <b>generalized enriched cyclic</b>            | <b>base framework</b>                             |

Two structural advantages distinguish the present theory. First, the effective convergence factor is governed by  $\varepsilon = (a + \theta)/(1 - a)$  rather than by a single contraction constant, which permits convergence in regimes where the underlying operator is not contractive and is only made so by averaging. Figure 12 compares the effective factor of the present framework with the corresponding factors of the Banach and Kannan principles as functions of the contractive parameter, showing that the admissible parameter range is strictly enlarged. Second, the point-dependent control function accommodates distance structures outside the reach of the metric and  $b$ -metric theories, while the convex structure makes the averaging construction available in spaces without linear operations. The combination of these features with the cyclic decomposition and with the generalized Reich condition is precisely what allows the single framework to encompass fixed point theory, fractal attractor theory, and the contraction-based analysis of physics-informed neural networks within one set of hypotheses.



**Fig. 12.** Effective convergence factor of the present framework against the Banach and Kannan factors as functions of the contractive parameter

## 7. Conclusion

This paper has developed a unified fixed point theory for generalized enriched cyclic contractions in convex controlled metric spaces. By coupling the Krasnoselskij averaging device with a cyclic representation and a point-dependent controlled triangle inequality, we established existence, uniqueness, and convergence theorems together with explicit a priori and a posteriori error estimates, and we obtained the Banach, Kannan, Reich, and Edelstein principles as particular cases governed by an effective convergence factor  $\varepsilon = (a + \theta)/(1 - a)$ . The analysis clarified two points that are easily mishandled in the controlled setting: the displacement term in the enriched Kannan condition contributes to the contraction factor and yields the sharp threshold  $\xi < \frac{1}{3}$  rather than disappearing, and the limiting arguments require continuity of the controlled distance together with a regularity device that supplies a telescopic Cauchy estimate. Building on the abstract theory, we constructed generalized iterated function systems driven by enriched cyclic contractions, analyzed the associated Hutchinson operator on the hyperspace endowed with the controlled Pompeiu–Hausdorff metric, and proved the existence and uniqueness of attractors, recovering the classical Cantor, Sierpiński, and fern fractals as concrete instances. Finally, we gave a physics-informed neural network realization in which the solution of an initial value problem is the unique fixed point of a Picard operator that is a Bielecki contraction, and hence a controlled-metric contraction, and a neural surrogate trained from scratch approximated that fixed point to an accuracy of order  $10^{-5}$ , in quantitative agreement with the contraction estimates. The framework thereby links fixed point theory, controlled metric geometry, fractal analysis, and scientific machine learning.

Several directions remain open, including best proximity results for the enriched cyclic class, multivalued and graph-directed generalized iterated function systems in the controlled setting, and the extension of the neural realization to systems of differential equations and to fractional models, where the controlled triangle inequality is expected to play a more pronounced role.

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## Conflicts of Interest

The authors declare no conflicts of interest.

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