



Solar Energy System Selection: A Pythagorean Fuzzy Rough Approach

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ABSTRACT

Existing approaches for multi-attribute decision-making rely solely on decision-makers' preferences and do not consider the degree of attribute prioritization. This paper introduces new prioritized aggregation operators within the Pythagorean fuzzy rough spectrum. These include the parameterized Schweitzer-Sklar t -norm and t -conorm to enhance variability in uncertain data and imprecision in applications. The Schweitzer-Sklar operators with adjustable parameters provide greater flexibility than traditional aggregation methods. Thus, the research presents a new class of prioritized aggregation operators in the Pythagorean fuzzy rough spectrum. The fundamental theoretical properties of these operators are thoroughly examined to verify their validity and potential uses. A novel multi-attribute decision-making approach using Pythagorean fuzzy rough sets to rank alternatives is developed. The proposed scientific approach is applied to a real-world solar energy system selection problem to demonstrate its practical utility and effectiveness. A comparison between the ranking results confirms the reliability and relevance of the proposed Pythagorean fuzzy rough approach.

1. Introduction

Recent years have seen a significant rise in interest in developing solar power systems due to the increasing demand for sustainable energy solutions. Choosing the best solar panels is a complex decision that involves considering several factors, including cost, performance under different environmental conditions, efficiency, and manufacturing materials.

In this paper, we introduce a new approach to address the uncertainties and complexities involved in assessing solar panels within the context of Pythagorean fuzzy rough (PyFR) Schweitzer-Sklar (SS) aggregation operators (AOs). Compared to traditional methods, the parameterized t -norm (TN) and t -conorm (TCN) of the SS framework offer greater flexibility in managing imprecise data. Decision-making (DMs) can better prioritize attributes and conduct a more thorough evaluation using the PyFR SS operators. This approach not only enhances the accuracy and reliability of the DM

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process but also provides a versatile tool for managing energy resources in real-world situations, especially amid uncertainty and changing priorities.

Zadeh [1] introduced the theory of fuzzy sets (FSs), which provided a method-based approach to dealing with uncertainties and anomalies in data and enables optimal decisions in the presence of uncertainty. FSs revolutionized traditional set theory by allowing elements to have a gradual membership degree (MD), which provided greater flexibility than the classical binary meanings. As a result, FSs have been widely applied in areas such as system control, decision analysis, and pattern recognition. Based on this fundamental idea, Atanassov [2] developed intuitionistic fuzzy sets (IFSs), which resulted in two types of functions: MD representing the number of inclusions and non-membership degree (NMD) representing the number of exclusions, with their sum equal to or less than 1. Grounded on the IFS theory, Yager [3] familiarized Pythagorean fuzzy sets (PyFSs) as a way to increase the expressiveness and flexibility of FSs and IFSs even further. In PyFSs, the sum of squares of MD and NMD is bound to fall inside the unit interval $[0, 1]$. In comparison to FSs and IFSs, this extension offers a wider range of membership configurations, making it a more potent tool for managing ambiguity and uncertainty in difficult DM situations, especially when determining DM requires more accuracy.

AOs play a crucial role in fuzzy DM by uniting individual input values, such as expert opinions or attribute scores, into a single representative value [4]. These operators help handle uncertainty and imprecision intrinsic to real-world problems [5]. Commonly used fuzzy AOs include TNs, TCNs, weighted averages, and ordered weighted averaging (OWA) operators [6]. Their flexibility and adaptability make them indispensable tools in multiple-attribute decision-making (MADM) [7]. Schweizer & Sklar [8] introduced the concept of integrated strategies, which provides a simple and efficient DM process. Several mathematicians have emphasized the reliability of the SS aggregation tool in dealing with ambiguous situations. For example, Garg [9] used Hamacher AOs to develop new clustering techniques under the IFS environment. Khan *et al.* [10] suggested the SS AO method, aiming to rigorously address the relationships between independent variables. Khan *et al.* [11] introduced new SS power Heronian mean AOs incorporating T-spherical fuzzy information. Ullah *et al.* [12] announced the prioritized Aczel-Alsina (AA) operators in the context of complex IFSs. Sarfraz [13] constructed an innovative approach utilizing T-spherical fuzzy information in conjunction with SS operators. Kausar *et al.* [14] focused on measuring the efficiency of the retrieval algorithm with SS information AOs. Ali *et al.* [15] developed a theoretical foundation for SS AOs based on neutrosophic fuzzy rough sets and their applications in green supply chain management. Wei *et al.* [16] extended the Fermatean fuzzy SS operators with applications in green supplier selection. Garg *et al.* [17] investigated the SS-based PAOs under an IFS environment with MADM applications.

Solar panel selection is a typical MADM problem because it comprises evaluating multiple conflicting attributes such as cost, efficiency, durability, environmental impact, and maintenance. A structured DM method helps classify the most appropriate panel by balancing these factors based on user preferences and practical restraints. Rani *et al.* [18] created the SWARA-VIKOR under the PyFS framework for solar panel selection. Suman *et al.* [19] articulated the MADM mechanism for the selection of photovoltaic cell technology via novel information measures under the PyFS environment. Li *et al.* [20] studied the evaluation of dispatching results of a power system with high penetration of renewable energy based on the PyFS and TOPSIS strategy. Dehshiri *et al.* [21] utilized the theory of PyFS on renewable energy projects based on sustainability goals. Mirzaei *et al.* [22] used a PyFS for a solar power plant location. Kumar & Mahanta [23] developed the concept of norm-based Pythagorean fuzzy metric and its application for solar panel selection. Al-Barakati *et al.* [24] proposed an extended interval-valued Pythagorean fuzzy WASPAS method based on new similarity measures to evaluate renewable energy sources. Ilbahar *et al.* [25] developed the concept of the

Pythagorean fuzzy WASPAS technique for renewable energy. Yin *et al.* [26] proposed a technique for site selection of a rail transit photovoltaic power station utilizing an interval-valued PyFS environment. Kamali Saraji *et al.* [27] framed an innovative fuzzy-SWARA-TOPSIS scheme with application towards energy development.

1.1 Motivation for the Study

In recent times, the selection and evaluation of solar panels have appeared as a critical MADM problem due to the growing global emphasis on sustainable and renewable energy sources. Nevertheless, this decision process is often characterized by a high degree of uncertainty, vagueness, and subjective judgment, which traditional crisp or classical fuzzy models struggle to handle effectively. To cope with these challenges, this study introduces a novel hybrid framework that integrates PyFRSs, SS TN, SS TCN, and PAOs for a more robust and realistic assessment of solar panel alternatives. The PyFRSs offer enhanced flexibility in modelling hesitation and uncertainty compared to IFSs, especially by allowing a wider range of MD and NMD. Meanwhile, the SS aggregation tools provide a generic and parametric class of operators capable of adapting to different interaction behaviors between attributes, such as compensatory and non-compensatory relationships. Furthermore, incorporating PAOs addresses the real-world dilemmas where not all attributes contribute equally to decision outcomes; i.e., some factors, like efficiency, cost, and durability, naturally carry more importance than others. Despite the advancements in fuzzy DM, prevailing studies often overlook this vital facet of attribute prioritization and depend on simplistic or equal-weight aggregation. Hence, this research fills a significant knowledge gap by proposing a comprehensive and adaptable MADM approach that not only manages uncertainty effectively but also reflects the varying importance of evaluation attributes. Eventually, this model aims to bridge the gap between theoretical development and practical application, providing a powerful tool for experts and policymakers involved in sustainable energy planning and solar technology assortment.

1.2 Contribution

The key contributions of the script are encapsulated as follows:

- i. To reflect attribute prioritization, this article hybridizes PAOs with SS operational laws within the PyFRS framework. This combination leads to the creation of new AOs, called the Schweitzer-Sklar prioritized averaging (PyFRSSPA), Pythagorean fuzzy rough Schweitzer-Sklar prioritized geometric (PyFRSSPG), Pythagorean fuzzy rough Schweitzer-Sklar weighted prioritized geometric (PyFRSSWPG), and Pythagorean fuzzy rough Schweitzer-Sklar weighted prioritized aggregation (PyFRSSWPA) operators;
- ii. We scrutinize the essential properties of the planned AOs;
- iii. We have framed an innovative MADM method grounded in newly established AOs tailored for PyFR information;
- iv. A case study related to the selection of solar panels is employed to demonstrate the efficacy and superiority of the suggested MADM strategy;
- v. A rigorous comparison and sensitivity analysis are accomplished to confirm the stability and robustness of the obtained results.

The rest of the article is structured as follows: In Section 2, we reviewed the basic terminologies related to PyFRS and their combination with SS functions. Section 3 presents the notions of PyFRSSPA,

PyFRSSWPA, PyFRSSPG, and PyFRSSWPG operators and analyzes their cardinal characteristics. In Section 4, we utilised the proposed operators to construct the MAGDM framework, which addresses uncertainty in DM dilemmas within the PyFRS environment. Also, a practical case study of solar panel selection is presented to legitimize the practicality of the proposed methodology. In Section 5, we analyse how changes in the SS parameter affect the results and compare our method with several other existing approaches to show how effective and suitable our technique is. We demonstrated the advantages of these operators, highlighting their superiority over existing methods. Section 6 consists of concluding remarks with future research perspectives.

2. Preliminaries

Some fundamental notions related to PyFSs, PyFRS, SS TN, SS TCN, and PAOs are described in this section.

Definition 1 [28]: A PyFS \bar{L} over the universal set \bar{U} is defined by:

$$\bar{L} = \{(\bar{Q}(X), \bar{E}(X)) : X \in \bar{U}\}, \quad (1)$$

where $\bar{Q}(X): \bar{L} \rightarrow [0, 1]$ and $\bar{E}(X): \bar{L} \rightarrow [0, 1]$ denote the MD and NMD of the element $X \in \bar{U}$ to the set \bar{L} , respectively, subjected to the constraint $0 \leq \bar{Q}^2(X) + \bar{E}^2(X) \leq 1$. Additionally, the degree of determinacy of X to the \bar{L} is stated as: $\bar{\delta}(X) = \sqrt{1 - (\bar{Q}^2(X) + \bar{E}^2(X))}$.

Definition 2 [29]: Let γ be a relation over the universal set Φ . Define a mapping $\Phi^*: \gamma \rightarrow \mathcal{A}(\gamma)$ as $\Phi^*(a) = \{M \in \gamma : (a, M) \in \Phi\}$ for $a \in \gamma$, where $\Phi^*(a)$ is known as the successor neighborhood of a concerning Φ and (γ, Φ) is said to be the crisp space of the approximation. For any subset $\alpha \subseteq \gamma$, the lower and upper approximations of α are respectively defined as:

$$\Phi^{LA}(\alpha) = \{M \in \Phi | \Phi^*(M) \subseteq \alpha\}, \quad (2)$$

$$\Phi^{UA}(\alpha) = \{M \in \Phi | \Phi^*(M) \cap \alpha \neq \emptyset\}, \quad (3)$$

where the set $(\Phi^{LA}(\alpha), \Phi^{UA}(\alpha))$ is regarded as a rough set (RS) regarding γ .

Definition 3 [29]: Let γ to be the universal set and Φ to be the relation over $PyFS(\gamma \times \gamma)$. Then:

- i. Φ is said to be reflexive if $Q_{\Phi}(M, M) = 1$ and $E_{\Phi}(M, M) = 0 \forall M \in \gamma$;
- ii. Φ is called symmetric if $\forall (M, \nabla) \in \gamma \times \gamma$, then $Q_{\Phi}(\nabla, M) = Q_{\Phi}(M, \nabla) \forall M, j \in \gamma$ and $E_{\Phi}(\nabla, M) = E_{\Phi}(M, \nabla)$;
- iii. Φ is called transitive when $\forall M, \nabla, \omega \in \gamma$ if $(j, \omega) \in \Phi$ and $(\omega, M) \in \Phi$, then we have $Q_{\Phi}(\nabla, M) \geq V[Q_{\Phi}(\nabla, \omega) \wedge Q_{\Phi}(\omega, M)]$, $E_{\Phi}(\nabla, M) \geq \wedge[E_{\Phi}(\nabla, \omega) \wedge E_{\Phi}(\omega, M)]$.

Definition 4: If we take γ to be the universal set and $\Phi \in PyFS(\gamma \times \gamma)$ to be the Pythagorean fuzzy (PyF) relation, then the PyF approximation space is an object of the form (γ, Φ) . The characterizations of the PyF lower and upper approximations for any subset $\alpha \subseteq \gamma$ are respectively postulated as:

$$\Phi^{PyFUA}(\alpha) = \{M, Q_{\Phi^{PyFUA}(\alpha)}(M), E_{\Phi^{PyFUA}(\alpha)}(M), |M \in \gamma\}, \quad (4)$$

$$\Phi^{\text{PyFLA}}(\alpha) = \{M, \mathbb{Q}_{\Phi^{\text{PyFLA}}(\alpha)}(M), \mathbb{E}_{\Phi^{\text{PyFLA}}(\alpha)}(M), |M \in \gamma\}, \quad (5)$$

where:

$$\left. \begin{aligned} \mathbb{Q}_{\Phi^{\text{PyFUA}}(\alpha)}(M) &= \bigvee_{l \in \gamma} [\mathbb{Q}_{\Phi(M)}(M, l) \vee \mathbb{Q}_{\alpha(M)}], \\ \mathbb{E}_{\Phi^{\text{PyFUA}}(\alpha)}(M) &= \bigwedge_{l \in \gamma} [\mathbb{E}_{\Phi(M)}(M, l) \wedge \mathbb{E}_{\alpha(M)}], \\ \mathbb{Q}_{\Phi^{\text{PyFLA}}(\alpha)}(M) &= \bigwedge_{l \in \gamma} [\mathbb{Q}_{\Phi(M)}(M, l) \wedge \mathbb{Q}_{\alpha(M)}], \\ \mathbb{E}_{\Phi^{\text{PyFLA}}(\alpha)}(M) &= \bigvee_{l \in \gamma} [\mathbb{E}_{\Phi(M)}(M, l) \vee \mathbb{E}_{\alpha(M)}], \end{aligned} \right\} \quad (6)$$

such that $0 \leq \mathbb{Q}_{\Phi^{\text{PyFUA}}(\alpha)}(M) + \mathbb{E}_{\Phi^{\text{PyFUA}}(\alpha)}(M) \leq 1$ and $0 \leq \mathbb{Q}_{\Phi^{\text{PyFLA}}(\alpha)}(M) + \mathbb{E}_{\Phi^{\text{PyFLA}}(\alpha)}(M) \leq 1$. Moreover, the set $(\Phi^{\text{PyFLA}}(\alpha), \Phi^{\text{PyFUA}}(\alpha))$ is termed as PyFRS. For convenience, $\bar{\mathcal{L}} = ((\bar{\mathbb{Q}}^L, \bar{\mathbb{E}}^L), (\bar{\mathbb{Q}}^U, \bar{\mathbb{E}}^U))$ is titled a PyF rough value (PyRFV).

Definition 5: For a PyFRV $\bar{\mathcal{L}} = ((\bar{\mathbb{Q}}^L, \bar{\mathbb{E}}^L), (\bar{\mathbb{Q}}^U, \bar{\mathbb{E}}^U))$, the score and accuracy functions are defined as:

$$\mathfrak{S}(\bar{\mathcal{L}}) = \frac{1 + (\bar{\mathbb{Q}}^L)^2 - (\bar{\mathbb{Q}}^U)^2 + (\bar{\mathbb{E}}^L)^2 - (\bar{\mathbb{E}}^U)^2}{2}, \quad (7)$$

$$\mathfrak{A}(\bar{\mathcal{L}}) = \frac{1 + (\bar{\mathbb{Q}}^L)^2 + (\bar{\mathbb{Q}}^U)^2 + (\bar{\mathbb{E}}^L)^2 + (\bar{\mathbb{E}}^U)^2}{2}, \quad (8)$$

where $\mathfrak{S}(\bar{\mathcal{L}}) \in [-1, 1]$ and $\mathfrak{A}(\bar{\mathcal{L}}) \in [0, 1]$ for all PyFVs.

Definition 6: The SS TN and SS TCN are articulated as:

$$\Psi_{SS}(x, y) = (x^\Delta + y^\Delta - 1)^{1/\Delta}, \quad (9)$$

$$\Psi^*_{SS}(x, y) = 1 - ((1-x)^\Delta + (1-y)^\Delta - 1)^{1/\Delta}, \quad (10)$$

where $\Delta < 0$ and $x, y \in [0, 1]$.

Definition 7 [28]: Let $\bar{\mathcal{L}}_j = ((\bar{\mathbb{Q}}_j^L, \bar{\mathbb{E}}_j^L), (\bar{\mathbb{Q}}_j^U, \bar{\mathbb{E}}_j^U))$ ($j = 1, 2$) be any two PyRFVs. Then, the SS operational rules are articulated as follows:

$$\bar{\mathcal{L}}_1 \oplus \bar{\mathcal{L}}_2 = \left(\left(\left(\sqrt{\left(1 - \left((1 - (\bar{\mathbb{Q}}_1^L)^2)^{\bar{\vartheta}} + (1 - (\bar{\mathbb{Q}}_2^L)^2)^{\bar{\vartheta}} \right) - 1 \right)^{\frac{1}{\bar{\vartheta}}}}, \sqrt{\left(((\bar{\mathbb{E}}_1^L)^2)^{\bar{\vartheta}} + ((\bar{\mathbb{E}}_2^L)^2)^{\bar{\vartheta}} - 1 \right)^{\frac{1}{\bar{\vartheta}}}} \right), \left(\left(\sqrt{\left(1 - \left((1 - (\bar{\mathbb{Q}}_1^U)^2)^{\bar{\vartheta}} + (1 - (\bar{\mathbb{Q}}_2^U)^2)^{\bar{\vartheta}} \right) - 1 \right)^{\frac{1}{\bar{\vartheta}}}}, \sqrt{\left(((\bar{\mathbb{E}}_1^U)^2)^{\bar{\vartheta}} + ((\bar{\mathbb{E}}_2^U)^2)^{\bar{\vartheta}} - 1 \right)^{\frac{1}{\bar{\vartheta}}}} \right) \right), \quad (11)$$

$$\overline{\mathcal{L}}_1 \otimes \overline{\mathcal{L}}_2 = \left(\left(\sqrt{\left(\left(\left(\overline{\mathcal{Q}}_1^L \right)^2 \right)^{\overline{\vartheta}} + \left(\left(\overline{\mathcal{Q}}_2^L \right)^2 \right)^{\overline{\vartheta}} \right) - 1} \right)^{\frac{1}{\overline{\vartheta}}}, \sqrt{\left(1 - \left(1 - \left(\overline{\mathcal{E}}_1^L \right)^2 \right)^{\overline{\vartheta}} + \left(1 - \left(\overline{\mathcal{E}}_2^L \right)^2 \right)^{\overline{\vartheta}} - 1 \right)^{\frac{1}{\overline{\vartheta}}} \right), \right. \quad (12)$$

$$\left. \left(\left(\sqrt{\left(\left(\left(\overline{\mathcal{Q}}_1^U \right)^2 \right)^{\overline{\vartheta}} + \left(\left(\overline{\mathcal{Q}}_2^U \right)^2 \right)^{\overline{\vartheta}} \right) - 1} \right)^{\frac{1}{\overline{\vartheta}}}, \sqrt{\left(1 - \left(1 - \left(\overline{\mathcal{E}}_1^U \right)^2 \right)^{\overline{\vartheta}} + \left(1 - \left(\overline{\mathcal{E}}_2^U \right)^2 \right)^{\overline{\vartheta}} - 1 \right)^{\frac{1}{\overline{\vartheta}}} \right) \right)$$

$$\Psi \overline{\mathcal{L}}_1 = \left(\left(\sqrt{\left(1 - \left(\Psi \left(1 - \left(\overline{\mathcal{Q}}_1^L \right)^2 \right)^{\overline{\vartheta}} - (\Psi - 1) \right) \right)^{\frac{1}{\overline{\vartheta}}}, \sqrt{\left(\Psi \left(1 - \left(\overline{\mathcal{E}}_1^L \right)^2 \right)^{\overline{\vartheta}} - (\Psi - 1) \right)^{\frac{1}{\overline{\vartheta}}} \right), \right. \quad (13)$$

$$\left. \left(\left(\sqrt{\left(1 - \left(\Psi \left(1 - \left(\overline{\mathcal{Q}}_1^U \right)^2 \right)^{\overline{\vartheta}} - (\Psi - 1) \right) \right)^{\frac{1}{\overline{\vartheta}}}, \sqrt{\left(\Psi \left(1 - \left(\overline{\mathcal{E}}_1^U \right)^2 \right)^{\overline{\vartheta}} - (\Psi - 1) \right)^{\frac{1}{\overline{\vartheta}}} \right) \right)$$

$$\overline{\mathcal{L}}_1^\Psi = \left(\left(\sqrt{\left(\left(\left(\Psi \left(1 - \left(\overline{\mathcal{Q}}_1^L \right)^2 \right)^{\overline{\vartheta}} - (\Psi - 1) \right) \right)^{\frac{1}{\overline{\vartheta}}}, \sqrt{\left(1 - \left(\Psi \left(1 - \left(\overline{\mathcal{E}}_1^L \right)^2 \right)^{\overline{\vartheta}} - (\Psi - 1) \right) \right)^{\frac{1}{\overline{\vartheta}}} \right) \right) \right. \quad (14)$$

$$\left. \left(\left(\sqrt{\left(\left(\left(\Psi \left(1 - \left(\overline{\mathcal{Q}}_1^U \right)^2 \right)^{\overline{\vartheta}} - (\Psi - 1) \right) \right)^{\frac{1}{\overline{\vartheta}}}, \sqrt{\left(1 - \left(\Psi \left(1 - \left(\overline{\mathcal{E}}_1^U \right)^2 \right)^{\overline{\vartheta}} - (\Psi - 1) \right) \right)^{\frac{1}{\overline{\vartheta}}} \right) \right) \right)$$

Definition 8: Let $\mathfrak{E} = \{d_1, d_2, \dots, d_n\}$ be a gathering of n attributes. Let there exist a priority relation between attributes $d_1 > d_2 > \dots > d_n$, which signifies an attribute c_i is superior to d_j if $i < j$. The value $d_i(t)$ indicates the performance of any alternative t concerning the attribute d_i such that $d_i(t) \in [0, 1]$. The expression described below is said to be the PAO:

$$PAO(d_i(t)) = \bigoplus_{i=1}^n \mathbb{P}_i d_i(t), \quad (15)$$

where $\mathbb{P}_i = \frac{w_i}{\sum_{i=1}^n w_i}$ with $\mathbb{P}_1 = 1$ and $\mathbb{P}_i = \prod_{k=1}^{i-1} d_k(t); i = 1, 2, \dots, n$.

3. Proposed Schweizer-Sklar Prioritized Aggregation Operators for PyFR Information

This section presents the systematic derivation of a novel class of PAOs based on SS TN and SS TCN within the framework of PyFSs. Specifically, we introduce and formulate the PyFRSSPA, PyFRSSWPA, PyFRSSPG, and PyFRSSWPG operators. Additionally, we carefully outline several key and beneficial features of these operators.

Definition 9: Let $\overline{\mathcal{L}}_j = \left((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U) \right)$ ($j = 2, 3, \dots, n$) be an array of PyFRVs. Then, the PyFRSSPA operator is stated as follows:

$$PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \mathbb{H}_1 \overline{\mathcal{L}}_1 \oplus \mathbb{H}_2 \overline{\mathcal{L}}_2 \oplus \dots \oplus \mathbb{H}_n \overline{\mathcal{L}}_n = \bigoplus_{j=1}^n \mathbb{H}_j \overline{\mathcal{L}}_j, \tag{16}$$

where $\mathbb{H}_j = \frac{\mathbb{H}_j}{\sum_{j=1}^n \mathbb{H}_j}$, $\mathbb{H}_1 = 1$, and $\mathbb{H}_j = \bigoplus_{\kappa=1}^{j-1} \mathfrak{S}(\overline{\mathcal{L}}_\kappa)$ $\kappa = 2, 3, \dots, n$. Here, $\mathfrak{S}(\overline{\mathcal{L}}_\kappa)$ denoted the score values k th PyFRV.

Theorem 1: For any collection of PyFRVs $\overline{\mathcal{L}}_j = \left((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U) \right)$, ($j = 1, 2, \dots, n$), the aggregated outcomes determined through the PyFRSSPA operator are still a PyFRV, which is given as:

$$PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \left(\left(\sqrt{1 - \left(\sum_{j=1}^n \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}}, \sqrt{\left(\sum_{j=1}^n \mathbb{H}_j \left((\overline{\mathbb{E}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right), \left(\sqrt{1 - \left(\sum_{j=1}^n \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}}, \sqrt{\left(\sum_{j=1}^n \mathbb{H}_j \left((\overline{\mathbb{E}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right). \tag{17}$$

Proof of Theorem 1 is provided in Appendix 1.

In the following, we examine some key features of the PyFRSSPA operator, including idempotency, monotonicity, and boundedness.

Theorem 2 (idempotency): Let $\overline{\mathcal{L}}_j = \left((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U) \right)$ ($j = 1, 2, \dots, n$) be an assemblage of PyFRVs. When $\overline{\mathcal{L}} = \overline{\mathcal{L}}_j \forall j$, then:

$$PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \overline{\mathcal{L}}. \tag{18}$$

Proof of Theorem 2 is provided in Appendix 2.

Theorem 3 (monotonicity): Presume that $\overline{\mathcal{L}}_j = \left((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U) \right)$ and $\overline{\mathcal{L}}'_j = \left((\overline{\mathbb{Q}}_j'^L, \overline{\mathbb{E}}_j'^L), (\overline{\mathbb{Q}}_j'^U, \overline{\mathbb{E}}_j'^U) \right)$ be two groups of PyFRVs such that $\overline{\mathcal{L}}_j \leq \overline{\mathcal{L}}'_j$, then:

$$PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq PyFRSSPA(\overline{\mathcal{L}}'_1, \overline{\mathcal{L}}'_2, \dots, \overline{\mathcal{L}}'_n). \tag{19}$$

Proof of Theorem 3 is provided in Appendix 3.

Theorem 4 (boundedness): Let $\mathcal{L}_j^- = \{\min \overline{\mathbb{Q}}_j^L, \max \overline{\mathbb{E}}_j^L\}$ and $\mathcal{L}_j^+ = \{\max \overline{\mathbb{Q}}_j^U, \min \overline{\mathbb{E}}_j^U\}$ be the smallest and largest PyFRVs, respectively. Then, we have:

$$\mathcal{L}_j^- \leq \text{PyFRSSPA}(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq \mathcal{L}_j^+. \tag{20}$$

Proof of Theorem 4 is provided in Appendix 4.

Definition 10: Presume that $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ ($j = 1, 2, \dots, n$) be a family of PyFRVs. Then, the PyFRSSWPA operator is expressed:

$$\text{PyFRSSWPA}(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \mathbb{H}_1 \overline{\mathcal{L}}_1 \oplus \mathbb{H}_2 \overline{\mathcal{L}}_2 \oplus \dots \oplus \mathbb{H}_n \overline{\mathcal{L}}_n = \bigoplus_{j=1}^n \mathbb{H}_j \overline{\mathcal{L}}_j, \tag{21}$$

where $\mathbb{H}_j = \frac{\varpi_j \mathbb{H}_j}{\sum_{j=1}^n \varpi_j \mathbb{H}_j}$, $\mathbb{H}_1 = 1$, and $\mathbb{H}_j = \bigoplus_{\kappa=1}^{j-1} \mathbb{S}(\overline{\mathcal{L}}_\kappa)$, $\kappa = 2, 3, \dots, n$. Moreover, ϖ_j represents the weight vector, such that $\varpi_j \in [0, 1]$ with $\sum_{j=1}^n \varpi_j = 1$.

Theorem 5: Let $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ ($j = 1, 2, \dots, n$) be a group of PyFRVs. Then, the accumulated outcome derived through the PyFRSSWPA operator is still a PyFRV and is described as:

$$\text{PyFRSSWPA}(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \left(\left(\begin{array}{c} \sqrt{\left(1 - \left(\sum_{j=1}^n \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{E}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right)^{\frac{1}{\overline{\vartheta}}} \\ \sqrt{\left(\sum_{j=1}^n \mathbb{H}_j \left((\overline{\mathbb{E}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \end{array} \right) \right)^{\frac{1}{\overline{\vartheta}}}. \tag{22}$$

Proof of Theorem 5 is similar to the proof of Theorem 1.

The PyFRSSWPA operator also fulfils the characteristics of idempotency, monotonicity, and boundedness.

Theorem 6 (idempotency): Let $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ ($j = 1, 2, \dots, n$) be a collection of PyFRVs. If $\overline{\mathcal{L}} = \overline{\mathcal{L}}_j \forall j$, then:

$$\text{PyFRSSWPA}(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \overline{\mathcal{L}}. \tag{23}$$

Proof of Theorem 6 is similar to the proof of Theorem 2.

Theorem 7 (monotonicity): Assume that $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ and $\overline{\mathcal{L}}_j' = ((\overline{\mathbb{Q}}_j'^L, \overline{\mathbb{E}}_j'^L), (\overline{\mathbb{Q}}_j'^U, \overline{\mathbb{E}}_j'^U))$ be two assemblages of PyFRVs such that $\overline{\mathcal{L}}_j \leq \overline{\mathcal{L}}_j'$. Then:

$$PyFRSSWPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq PyFRSSWPA(\overline{\mathcal{L}}_1^-, \overline{\mathcal{L}}_2^-, \dots, \overline{\mathcal{L}}_n^-). \quad (24)$$

Proof of Theorem 7 is similar to the proof of Theorem 3.

Theorem 8 (boundedness): Suppose that $\mathcal{L}_j^- = \{\min \overline{\mathbb{Q}}_j^L, \max \overline{\mathbb{E}}_j^L\}$ and $\mathcal{L}_j^+ = \{\max \overline{\mathbb{Q}}_j^U, \min \overline{\mathbb{E}}_j^U\}$ be the smallest and largest PyFRVs, respectively. Then, we have:

$$\mathcal{L}_j^- \leq PyFRSSWPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq \mathcal{L}_j^+. \quad (25)$$

Proof of Theorem 8 is similar to the proof of Theorem 4.

Definition 11: Let $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ ($j = 2, 3, \dots, n$) be a collection of PyFRVs. Then, the PyFRSSPG operator is articulated as follows:

$$PyFRSSPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \overline{\mathcal{L}}_1^{\mathbb{H}_1} \otimes \overline{\mathcal{L}}_2^{\mathbb{H}_2} \dots \overline{\mathcal{L}}_n^{\mathbb{H}_n} = \otimes_{j=1}^n \overline{\mathcal{L}}_j^{\mathbb{H}_j}, \quad (26)$$

where $\mathbb{H}_j = \frac{\mathbb{H}_j}{\sum_{j=1}^n \mathbb{H}_j}$, $\mathbb{H}_1 = 1$, and $\mathbb{H}_j = \bigoplus_{\kappa=1}^{j-1} \mathfrak{S}(\overline{\mathcal{L}}_{\kappa}) \kappa = 2, 3, \dots, n$.

Theorem 9: Presume that $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ ($j = 1, 2, \dots, n$). Then, the aggregated result acquired via the PyFRSSPG operator is still a PyFRV, which is given as:

$$PyFRSSPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \left(\left(\left(\sqrt{1 - \left(\sum_{j=1}^n \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}}, \right. \right. \right. \quad (27)$$

$$\left. \left. \left. \sqrt{\left(\sum_{j=1}^n \mathbb{H}_j \left((\overline{\mathbb{E}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \right.$$

$$\left. \left(\left(\sqrt{1 - \left(\sum_{j=1}^n \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}}, \right. \right. \right.$$

$$\left. \left. \left. \sqrt{\left(\sum_{j=1}^n \mathbb{H}_j \left((\overline{\mathbb{E}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \right)$$

Proof of Theorem 9 is similar to the proof of Theorem 1.

From Theorem 9, it is apparent that the PyFRSSPG operator possesses the characteristics of idempotency, monotonicity, and boundedness, which can be described below.

Theorem 10 (idempotency): Let $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ ($j = 1, 2, \dots, n$) be an assembly of PyFRVs. If $\overline{\mathcal{L}} = \overline{\mathcal{L}}_j \forall j$, then:

$$PyFRSSPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \overline{\mathcal{L}}. \quad (28)$$

Proof of Theorem 10 is similar to the proof of Theorem 2.

Theorem 11 (monotonicity): Imagine that $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}^L, \overline{\mathbb{E}}^L), (\overline{\mathbb{Q}}^U, \overline{\mathbb{E}}^U))$ and $\overline{\mathcal{L}}'_j = ((\overline{\mathbb{Q}}'^L, \overline{\mathbb{E}}'^L), (\overline{\mathbb{Q}}'^U, \overline{\mathbb{E}}'^U))$ be two collections of PyFRVs such that $\overline{\mathcal{L}}_j \leq \overline{\mathcal{L}}'_j$. Then:

$$PyFRSSPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq PyFRSSPG(\overline{\mathcal{L}}'_1, \overline{\mathcal{L}}'_2, \dots, \overline{\mathcal{L}}'_n). \tag{29}$$

Proof of Theorem 11 is similar to the proof of Theorem 3.

Theorem 12 (boundedness): Let $\mathcal{L}_j^- = \{\min \overline{\mathbb{Q}}_j^L, \max \overline{\mathbb{E}}_j^L\}$ and $\mathcal{L}_j^+ = \{\max \overline{\mathbb{Q}}_j^U, \min \overline{\mathbb{E}}_j^U\}$ be the smallest and largest PyFRVs, respectively. Then:

$$\mathcal{L}_j^- \leq PyFRSSPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq \mathcal{L}_j^+. \tag{30}$$

Proof of Theorem 12 is similar to the proof of Theorem 4.

Definition 12: For a group of PyFRVs $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ ($j = 1, 2, \dots, n$), the notion of the PyFRSSWPG operator is stated as follows:

$$PyFRSSWPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \overline{\mathcal{L}}_1^{\mathbb{H}_1} \otimes \overline{\mathcal{L}}_2^{\mathbb{H}_2} \otimes \dots \otimes \overline{\mathcal{L}}_n^{\mathbb{H}_n} = \otimes_{j=1}^n \overline{\mathcal{L}}_j^{\mathbb{H}_j}, \tag{31}$$

where $\mathbb{H}_j = \frac{\varpi_j \mathbb{H}_j}{\sum_{j=1}^n \varpi_j \mathbb{H}_j}$, $\mathbb{H}_1 = 1$, and $\mathbb{H}_j = \bigoplus_{\kappa=1}^{j-1} \mathfrak{S}(\overline{\mathcal{L}}_\kappa)$, $\kappa = 2, 3, \dots, n$. Additionally, ϖ_j denotes the weight of the j th PyFRV such that $\varpi_j \in [0, 1]$ with $\sum_{j=1}^n \varpi_j = 1$.

Theorem 13: Let $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ ($j = 1, 2, \dots, n$) be a family of PyFRVs. Then, the fused outcome determined through the PyFRSSWPA operator is again a PyFRV, which is given as:

$$PyFRSSWPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \left(\left(\left(\sqrt{1 - \left(\sum_{j=1}^n \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}}, \right. \right. \right. \tag{32}$$

Proof of Theorem 13 is similar to the proof of Theorem 1.

Theorem 14 (idempotency): Assume that $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U))$ ($j = 1, 2, \dots, n$) be an assemblage of PyFRVs. If $\overline{\mathcal{L}} = \overline{\mathcal{L}}_j \forall j$, then:

$$PyFRSSWPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) = \overline{\mathcal{L}}. \tag{33}$$

Proof of Theorem 14 is similar to the proof of Theorem 2.

Theorem 15 (monotonicity): Suppose that $\overline{\mathcal{L}}_j = ((\overline{\mathbb{Q}}^L, \overline{\mathbb{E}}^L), (\overline{\mathbb{Q}}^U, \overline{\mathbb{E}}^U))$ and $\overline{\mathcal{L}}'_j = ((\overline{\mathbb{Q}}'^L, \overline{\mathbb{E}}'^L), (\overline{\mathbb{Q}}'^U, \overline{\mathbb{E}}'^U))$ be two gatherings of PyFRVs such that $\overline{\mathcal{L}}_j \leq \overline{\mathcal{L}}'_j$. Then:

$$PyFRSSWPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq PyFRSSWPG(\overline{\mathcal{L}}'_1, \overline{\mathcal{L}}'_2, \dots, \overline{\mathcal{L}}'_n). \quad (34)$$

Proof of Theorem 15 is similar to the proof of Theorem 3.

Theorem 16 (boundedness): Suppose that $\mathcal{L}_j^- = \{\min \overline{\mathbb{Q}}_j^L, \max \overline{\mathbb{E}}_j^L\}$ and $\mathcal{L}_j^+ = \{\max \overline{\mathbb{Q}}_j^U, \min \overline{\mathbb{E}}_j^U\}$ be the smallest and largest PyFRVs, respectively. Then:

$$\mathcal{L}_j^- \leq PyFRSSWPG(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq \mathcal{L}_j^+. \quad (35)$$

Proof of Theorem 16 is similar to the proof of Theorem 4.

4. Multiple-Attribute Decision-Making Methods for PyFR Information

The MADM process is essential for selecting the most suitable alternative from a set of options, especially when evaluations are provided by experts across multiple attributes. In this setup, each alternative is assessed based on a common set of criteria, with expert judgment represented using PyFRVs. This evaluation is organized into decision matrices and then combined, considering the relative importance (weights) of each expert's input. The weighted attributes are used to improve the overall evaluations of each alternative. Due to its robustness and versatility, MADM is widely used in various fields, including engineering, economics, business, and applied mathematics. In the current section, we aim to use PyFR information to build an MADM approach based on PyFRSSPA, PyFRSSWPA, PyFRSSPG, and PyFRSSWPG operators.

4.1 Problem Statement

Consider a discrete set of alternatives $\Omega = \{Q_1, Q_2, \dots, Q_m\}$ being evaluated through a finite collection of attributes denoted as $\mathcal{E} = \{\overline{e}_1, \overline{e}_2, \dots, \overline{e}_n\}$ that fulfils the condition of prioritization $\overline{e}_1 > \overline{e}_2 > \dots > \overline{e}_n$, which tells that the attribute \overline{e}_i has a higher priority than $\overline{e}_j, \forall i < j$. Let us presume that $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the related weight vector of attributes subjected to the constraint $\omega_j \in [0, 1]$ with $\sum_{j=1}^n \omega_j = 1$. Each alternative $Q_i (i = 1, 2, \dots, m)$ is assessed regarding the attributes $\overline{e}_j (j = 1, 2, \dots, n)$ by an expert. The assessment values are provided in the form of PyFRVs, characterized as $\overline{\mathcal{L}}_{ij} = ((\overline{\mathbb{Q}}_{ij}^L, \overline{\mathbb{E}}_{ij}^L), (\overline{\mathbb{Q}}_{ij}^U, \overline{\mathbb{E}}_{ij}^U))$. These values measure how the alternative Q_i satisfies the attribute \overline{e}_j fulfilling the constraints $0 \leq (\overline{\mathbb{Q}}_{ij}^L)^2 + (\overline{\mathbb{E}}_{ij}^L)^2 \leq 1$ and $0 \leq (\overline{\mathbb{Q}}_{ij}^U)^2 + (\overline{\mathbb{E}}_{ij}^U)^2 \leq 1$.

4.2 Decision Algorithm

The detailed step-by-step computational procedure of the proposed MADM framework is outlined as follows:

Step 1: The expert assessments for each alternative, related to its specific attributes, are organized in the PyFR decision matrix shown below:

$$M = [\overline{\mathcal{L}}_{ij}]_{m \times n} = [((\overline{\mathbb{Q}}_{ij}^L, \overline{\mathbb{E}}_{ij}^L), (\overline{\mathbb{Q}}_{ij}^U, \overline{\mathbb{E}}_{ij}^U))]_{m \times n}. \quad (36)$$

Step 2: Within the context of MADM paradigms, attributes are characteristically classified into two types: benefit-type and cost-type. To ensure comparability and eliminate the effect of differing attribute natures, it is crucial to normalize all attributes by converting them into a consistent form. The normalized decision matrix, symbolized as $N = [\overline{\overline{\mathcal{L}_{ij}}}]_{m \times n}$, is attained using the subsequent formula:

$$\overline{\overline{\mathcal{L}_{ij}}} = \begin{cases} \overline{\overline{\mathcal{L}_{ij}}} = ((\overline{\overline{Q_j^L}}, \overline{\overline{E_j^L}}), (\overline{\overline{Q_j^U}}, \overline{\overline{E_j^U}})); & \text{for benefit attribute} \\ \overline{\overline{\mathcal{L}_{ij}}}^c = ((\overline{\overline{E_j^L}}, \overline{\overline{Q_j^L}}), (\overline{\overline{E_j^U}}, \overline{\overline{Q_j^U}})); & \text{for cost attribute} \end{cases} \quad (37)$$

Step 3: Determine the prioritized values $\mathbb{H}_j = \frac{\mathbb{H}_j}{\sum_{j=1}^n \mathbb{H}_j}$, where $\mathbb{H}_1 = 1$ and $\mathbb{H}_j = \bigoplus_{\kappa=1}^{j-1} \mathfrak{S}(\overline{\overline{\mathcal{L}_\kappa})}$, $\kappa = 2, 3, \dots, n$.

Step 4: Compile the entire set of data and calculate the aggregated PyFRVs of $\overline{\overline{\mathcal{L}_{ij}}}$ related to $Q_i (i = 1, 2, \dots, m)$ concerning attributes $\overline{e}_j (j = 1, 2, \dots, n)$ using the PyFRSSPA, PyFRSSWPA, PyFRSSPG, and PyFRSSWPG operators.

Step 5: Calculate the score values of the aggregated PyFRVs attained in Step 4.

Step 6: Rank the alternatives $Q_i (i = 1, 2, \dots, m)$ in descending order based on their calculated score values, and classify the most preferred alternative as the optimal candidate.

The procedural flow of the proposed MADM scheme is portrayed in Figure 1.

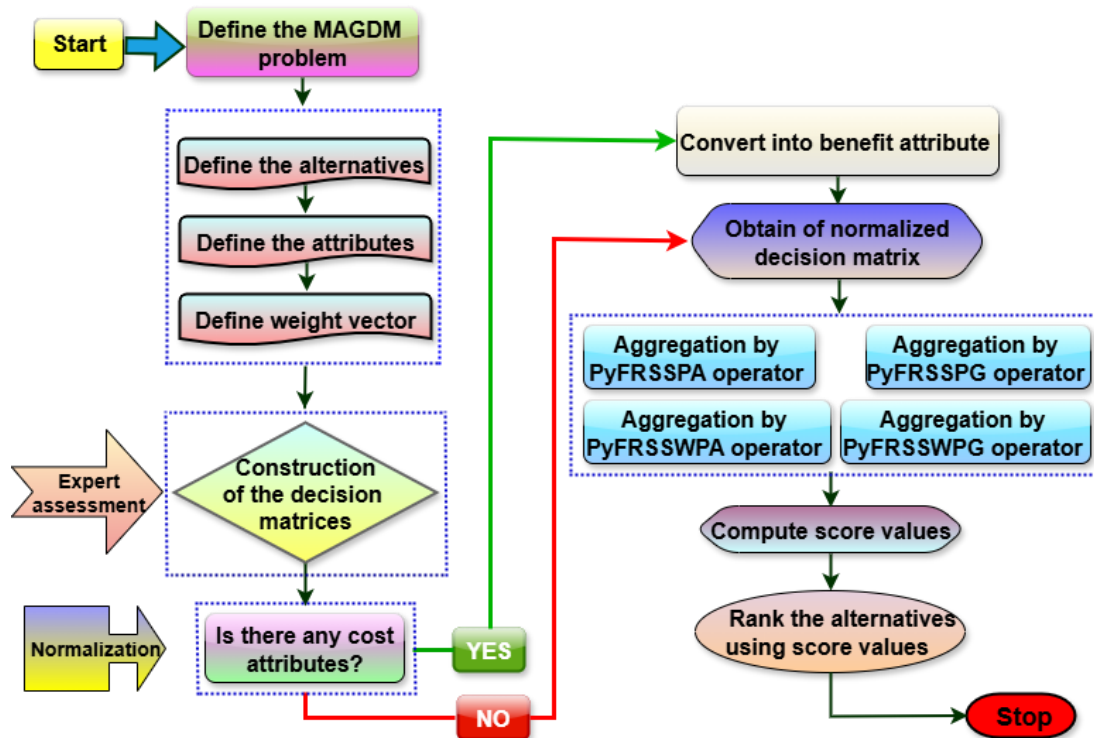


Fig. 1. Flowchart of proposed MADM method

4.3 Numerical Example: Solar Energy System Selection

The role of energy in promoting economic growth and social progress is clear, especially in emerging economies where electricity demand is rising quickly across industrial, agricultural,

healthcare, and residential sectors. Among different energy sources, solar energy has gained significant interest because of its sustainability, affordability, and environmental benefits. Considering the country's geographical location, with abundant sunlight year-round, solar energy provides a practical solution to address electricity shortages and maintain a reliable power supply, particularly for critical facilities like hospitals.

In this application, we consider the applied problem of choosing the most appropriate solar panel for a public hospital, aiming to enhance energy independence and operational reliability. The hospital must ensure that its energy source is reliable, mainly during prolonged power outages. Due to the presence of multiple alternatives and evaluation attributes with intrinsic uncertainty and vagueness, a robust MADM scheme is essential.

To validate the utility of the projected PyFRSSPA, PyFRSSWPA, PyFRSSPG, and PyFRSSWPG operators, a DM scenario is built with five solar panel alternatives $\mathcal{Q} = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$ assessed against four key attributes $\mathcal{C} = \{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$, outlined as follows:

- i. type of raw material (polycrystalline and monocrystalline materials) (\bar{e}_1);
- ii. panel quality and manufacturing standards (\bar{e}_2);
- iii. performance under low sunlight circumstances (\bar{e}_3);
- iv. duration of warranty (\bar{e}_4).

Each alternative's performance on these attributes is represented using PyFRVs, apprehending both epistemic uncertainty and expert hesitation. The decision expert provides evaluations and assigns weights to attributes based on their importance in the assortment setting, which is $w = (0.3, 0.35, 0.15, 0.2)^T$. To handle the attribute diversity (benefit vs. cost types), normalization is applied. The priority weights of alternatives are then determined via the proposed operators.

4.4 Decision Process

The methodical process of choosing an appropriate solar panel is explained as follows:

Step 1: The expert's evaluation data for the options being considered, based on the specific attributes, is organized in a decision matrix using PyFRFVs, as revealed in Table 1.

Table 1
 Expert assessment

	\bar{e}_1				\bar{e}_2				\bar{e}_3				\bar{e}_4			
	Q^L	E^L	Q^U	E^U	Q^L	E^L	Q^U	E^U	Q^L	E^L	Q^U	E^U	Q^L	E^L	Q^U	E^U
Q_1	0.35	0.26	0.37	0.34	0.28	0.37	0.32	0.46	0.24	0.41	0.27	0.45	0.32	0.28	0.42	0.44
Q_2	0.22	0.24	0.26	0.35	0.25	0.35	0.37	0.48	0.23	0.37	0.28	0.42	0.33	0.27	0.35	0.35
Q_3	0.25	0.21	0.33	0.32	0.34	0.37	0.38	0.45	0.21	0.38	0.25	0.44	0.34	0.37	0.36	0.45
Q_4	0.27	0.25	0.34	0.37	0.35	0.35	0.42	0.42	0.24	0.33	0.35	0.52	0.31	0.38	0.28	0.46
Q_5	0.34	0.23	0.42	0.43	0.31	0.33	0.46	0.48	0.35	0.34	0.42	0.44	0.25	0.35	0.42	0.44

Step 2: Since all the attributes under consideration are of the benefit type, normalization is not required in this case.

Step 3: We determine the prioritized values (Table 2) and prioritized weighted values (Table 3).

Table 2
 Prioritized values

	Score values				\bar{E}_j				Sum	\bar{e}_τ			
Q_1	0.469	0.451	0.475	0.405	1.0	0.469	0.211	0.100	1.780	0.562	0.263	0.119	0.056
Q_2	0.458	0.409	0.468	0.468	1.0	0.458	0.187	0.088	1.733	0.577	0.264	0.108	0.051
Q_3	0.448	0.453	0.466	0.460	1.0	0.448	0.203	0.094	1.745	0.573	0.257	0.116	0.054
Q_4	0.441	0.446	0.387	0.475	1.0	0.441	0.197	0.076	1.715	0.583	0.257	0.115	0.044
Q_5	0.404	0.382	0.434	0.408	1.0	0.404	0.154	0.067	1.624	0.616	0.248	0.095	0.041

Table 3
 Prioritized weighted values

	$\omega_\tau \bar{e}_\tau$					Sum	\bar{e}_τ				
Q_1	0.150	0.164	0.042	0.030	0.386	0.388	0.425	0.109	0.078		
Q_2	0.150	0.160	0.037	0.026	0.374	0.401	0.429	0.100	0.070		
Q_3	0.150	0.157	0.041	0.028	0.376	0.399	0.417	0.108	0.075		
Q_4	0.150	0.155	0.039	0.023	0.367	0.409	0.421	0.107	0.062		
Q_5	0.150	0.141	0.031	0.020	0.342	0.438	0.413	0.090	0.059		

Step 4: According to PyFRSSPA, PyFRSSWPA, PyFRSSPG, and PyFRSSWPG operators, aggregated PyFRVs regarding the alternatives $Q_1(1, 2, \dots, 5)$ about attributes $\bar{e}_j(j = 1, 2, 3, 4)$ are compiled in Table 4.

Table 4
 Aggregated PyFRVs

	PyFRSSPA				PyFRSSPG				PyFRSSPWA				PyFRSSPWG			
Q_1	0.321	0.351	0.293	0.379	0.308	0.341	0.316	0.398	0.338	0.394	0.261	0.342	0.243	0.273	0.388	0.453
Q_2	0.236	0.302	0.271	0.381	0.231	0.286	0.292	0.399	0.292	0.360	0.241	0.351	0.221	0.261	0.349	0.473
Q_3	0.279	0.338	0.249	0.360	0.263	0.328	0.295	0.383	0.331	0.371	0.211	0.321	0.212	0.253	0.367	0.444
Q_4	0.292	0.362	0.280	0.397	0.282	0.353	0.296	0.410	0.339	0.411	0.251	0.372	0.243	0.284	0.353	0.504
Q_5	0.331	0.431	0.258	0.442	0.327	0.429	0.277	0.445	0.335	0.452	0.231	0.432	0.254	0.421	0.328	0.473

Step 5: The scores of each alternative are displayed in Table 5.

Table 5
 Score values

	PyFRSSPA	PyFRSSPG	PyFRSSPWA	PyFRSSPWG
$\bar{S}(Q_1)$	0.498	0.476	0.480	0.458
$\bar{S}(Q_2)$	0.464	0.445	0.461	0.438
$\bar{S}(Q_3)$	0.500	0.471	0.494	0.461
$\bar{S}(Q_4)$	0.491	0.474	0.491	0.472
$\bar{S}(Q_5)$	0.516	0.508	0.509	0.50

Step 6: Based on the score values, the ranking results are given in Table 6.

Table 6
 Ranking orders

Operators	Ranking
PyFRSSPA	$Q_5 \succcurlyeq Q_3 \succcurlyeq Q_1 \succcurlyeq Q_4 \succcurlyeq Q_2$
PyFRSSPG	$Q_5 \succcurlyeq Q_1 \succcurlyeq Q_4 \succcurlyeq Q_3 \succcurlyeq Q_2$
PyFRSSPWA	$Q_5 \succcurlyeq Q_3 \succcurlyeq Q_4 \succcurlyeq Q_1 \succcurlyeq Q_2$
PyFRSSPWG	$Q_5 \succcurlyeq Q_4 \succcurlyeq Q_3 \succcurlyeq Q_1 \succcurlyeq Q_2$

According to Table 6, even though the rankings vary among the four different PyFR SS-based aggregation methods, the highest-ranked alternative and the lowest-ranked alternative stay consistent in all cases. This consistency shows the robustness of the planned approach in identifying the best alternative, despite differences in aggregation strategies. Finally, the graphical representation of the ranking results obtained through the formulated operators is given in Figure 2.

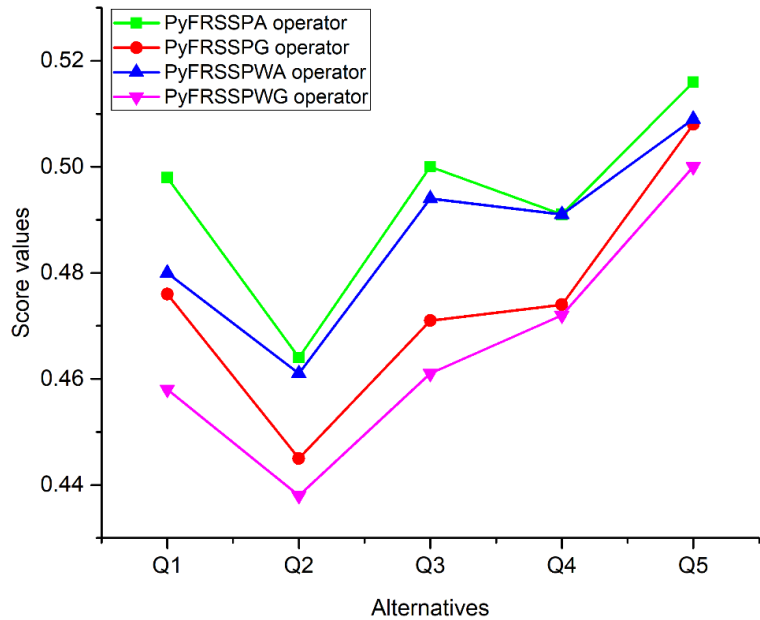


Fig. 2. Ranking outcomes via the proposed aggregation operators

5. Discussion

This section conducts a systematic sensitivity and comparative analysis to evaluate the robustness of the planned methodological framework. It is then followed by a detailed performance assessment through comparisons with several established methods.

5.1 Sensitivity Analysis

The SS TN and SS TCN models integrate a regulatory parameter, ϑ , which significantly enhances their adaptability and effectiveness in handling ambiguous information. This tunable flexibility permits better management of uncertainty, making these models particularly valuable in situations necessitating robust fuzzy data processing.

A comprehensive sensitivity analysis is conducted to estimate how the projected PyFRSSWPA and PyFRSSWPG operators respond to variations in ϑ . Specifically, the parameter ϑ is systematically varied within the range $[-1, -100]$, and its impact on the resulting score values and ranking orders is examined in detail. The computed ranking variations derived from the PyFRSSWPA and PyFRSSWPG operators are graphically presented in Figure 3 and Figure 4, respectively.

According to Figure 3, it becomes manifest that as the SS parameter decreases from -1 to -100, the corresponding score values of all alternatives display a consistent increasing trend in the environment of the PyFRSSWPA operator. This behaviour specifies that the PyFRSSWPA operator becomes more optimistic in assessing alternatives as the SS parameter becomes more negative, effectively enhancing its sensitivity to favourable information. Although the ranking order experiences slight variations across the range of the SS parameter, the alternatives Q_3 and Q_5

consistently appear among the top ranks, signifying the operator’s relative stability and robustness. This stability makes the PyFRSSWPA operator especially good for circumstances where it is important to clearly distinguish between options, even when there is a lot of uncertainty.

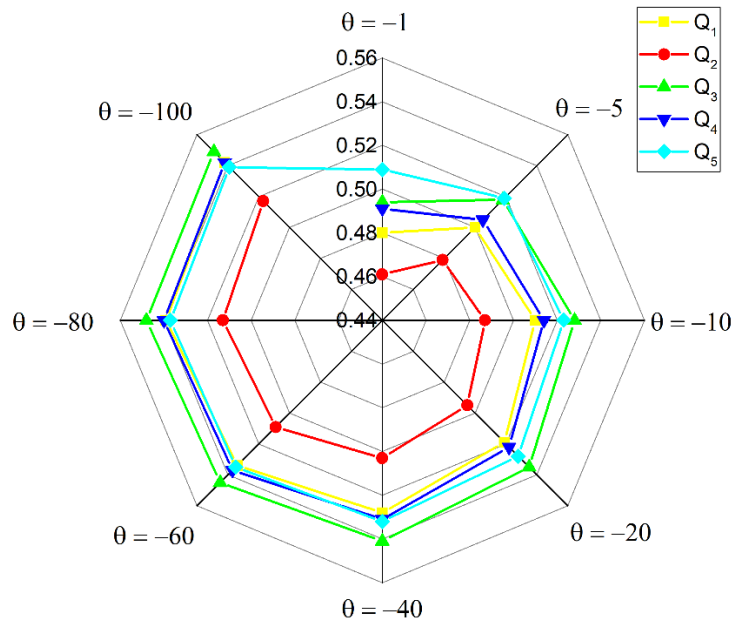


Fig. 3. Sensitivity analysis of θ under the PyFRSSWPA operator

Analogously, in Figure 4, we observed that the score values exhibit a decreasing trend with increasingly negative values of the parameter, indicating a more conservative aggregation behaviour. Despite the significant variation in scores, the ranking order remains relatively stable across all examined parameter values, with Q_5 persisting in its position as a top-ranked candidate. This high degree of ranking consistency underscores the robustness of the PyFRSSWPG operator and suggests its effectiveness in DM scenarios where maintaining a fixed ranking structure under uncertain or imprecise conditions is crucial.

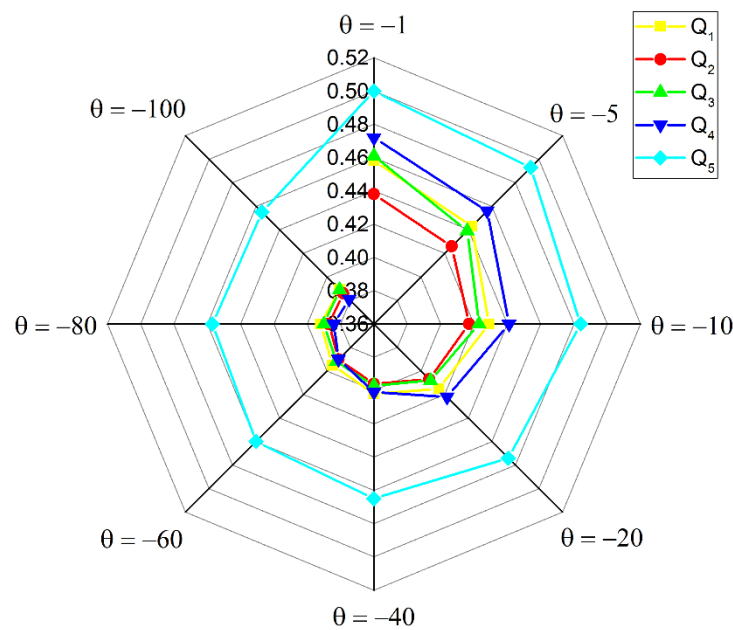


Fig. 4. Sensitivity analysis of θ under the PyFRSSWPG operator

5.2 Comparison with Existing Methods

In this part, we thoroughly compare the proposed operators and the MADM method with many other existing aggregation-based methods that use different AOs. Suman *et al.* [19] developed an MCDM-based framework for selecting photovoltaic cell technology using novel information measures under a PyF environment. Garg *et al.* [17] created the idea of IF rough SS weighted averaging (IFRSSWA) and IF rough SS weighted geometric (IFRSSWG) operators. Hussain *et al.* [30] created a DM method that uses PyF rough SS weighted averaging (PyFRSSWA) and PyF rough SS weighted geometric (PyFRSSWG) operators. In this regard, we resolve the formerly designated MADM problem with the weight vector $w = (0.3, 0.35, 0.15, 0.2)^T$ and letting $\theta = -1$. Table 7 compiles the resulting score values and ranking outcomes from the planned approach, along with those from the existing approaches.

Table 7
 Comparison analysis with some prevailing methods

Methods	Score values					Ranking
	$\tilde{S}(Q_1)$	$\tilde{S}(Q_2)$	$\tilde{S}(Q_3)$	$\tilde{S}(Q_4)$	$\tilde{S}(Q_5)$	
PyFRSSPA	0.4980	0.4640	0.50	0.4910	0.5160	$Q_5 \succcurlyeq Q_3 \succcurlyeq Q_1 \succcurlyeq Q_4 \succcurlyeq Q_2$
PyFRSSPG	0.4760	0.4450	0.4710	0.4740	0.5080	$Q_5 \succcurlyeq Q_1 \succcurlyeq Q_4 \succcurlyeq Q_3 \succcurlyeq Q_2$
PyFRSSPWA	0.480	0.4610	0.4940	0.4910	0.509	$Q_5 \succcurlyeq Q_3 \succcurlyeq Q_4 \succcurlyeq Q_1 \succcurlyeq Q_2$
PyFRSSPWG	0.4580	0.4380	0.4160	0.4720	0.50	$Q_5 \succcurlyeq Q_4 \succcurlyeq Q_3 \succcurlyeq Q_1 \succcurlyeq Q_2$
Suman <i>et al.</i> [19]	0.4543	0.3467	0.1426	0.1782	0.5229	$Q_5 \succcurlyeq Q_1 \succcurlyeq Q_2 \succcurlyeq Q_4 \succcurlyeq Q_3$
IFRSSWA [17]	0.3785	0.2453	0.0563	0.1745	0.3842	$Q_5 \succcurlyeq Q_1 \succcurlyeq Q_2 \succcurlyeq Q_4 \succcurlyeq Q_3$
IFRSSWG [17]	0.4740	0.4563	0.2672	0.1994	0.5832	$Q_5 \succcurlyeq Q_1 \succcurlyeq Q_2 \succcurlyeq Q_3 \succcurlyeq Q_4$
PyFRSSWA [30]	0.4745	0.2712	0.1672	0.0862	0.5732	$Q_5 \succcurlyeq Q_1 \succcurlyeq Q_2 \succcurlyeq Q_3 \succcurlyeq Q_4$
PyFRSSWG [30]	0.5074	0.4563	0.2512	0.0952	0.5742	$Q_5 \succcurlyeq Q_1 \succcurlyeq Q_2 \succcurlyeq Q_3 \succcurlyeq Q_4$

Table 7 indicates that the best option, Q_5 , found using all four recommended methods, matches up with the existing methods. The PyFRSSPWA operator shows a balanced score distribution with moderate discrimination among alternatives, resulting in a ranking of $Q_5 \succcurlyeq Q_3 \succcurlyeq Q_4 \succcurlyeq Q_1 \succcurlyeq Q_2$, highlighting its effectiveness in distinguishing between closely competitive options. On the other hand, the PyFRSSPWG operator demonstrates a more conservative ranking pattern but maintains stability by preserving Q_5 as the top-ranked alternative across all models. In contrast, traditional methods assign lower scores, particularly to options Q_3 and Q_4 , which significantly affects their rankings. For instance, Suman *et al.* [19] place Q_3 at the lowest rank, despite its competitive position in the proposed models, indicating weaker discriminatory power under uncertainty. Furthermore, operators such as PyFRSSWA and PyFRSSWG show significant variations in their scores, particularly underestimating alternatives like Q_4 and Q_3 , which could result in suboptimal decisions in real-world situations. These findings indicate that the proposed PyFR SS-based operators work better, are more dependable, and can successfully tell the difference between choices, showing they are valuable in fuzzy DM circumstances that require a lot of trust and careful evaluation.

6. Concluding Remarks

PAOs are effective tools for managing uncertainty in complex DM environments. At the same time, the PyFRS model provides a strong and adaptable way to handle different types of uncertainty in DM situations. In this study, we propose a novel MADM approach that integrates newly developed AOs, leveraging the strengths of SS operations, PAOs, and the PyFRS model. We outline the principal contributions of this research as follows:

- i. This study builds on the SS operations in the PyFRS context and adds new types of AOs, specifically the PyFRSSPA, PyFRSSWPA, PyFRSSPG, and PyFRSSWPG operators. A distinguishing advantage of these operators is their ability to incorporate attribute prioritization, which enhances DM precision compared to existing models;
- ii. The basic principles of the new AOs are carefully defined by examining important features like idempotency, boundedness, and monotonicity;
- iii. Using the new operators we created, we develop a new method for making decisions that allows for better and more adaptable rankings of options in uncertain situations;
- iv. The usefulness of the suggested MADM method is shown through a real-life example of evaluating solar panels, proving it works well in complicated situations with uncertainty;
- v. To assess the stability and robustness of the proposed method, we carry out an in-depth sensitivity analysis that evaluates how the SS parameter influences the final decision outcomes;
- vi. A comparative analysis is conducted against several existing techniques, affirming the superiority and effectiveness of the proposed framework in handling prioritized and uncertain information in MADM problems.

To broaden the application range of the planned methodology, future studies may explore the following perspectives:

- i. The anticipated methodology may be further extended by incorporating alternative uncertainty modelling frameworks, such as intuitionistic fuzzy rough sets, neutrosophic sets, or q-rung orthopair fuzzy settings, to enhance its applicability across a wider range of real-world dilemmas;
- ii. Future studies could explore the incorporation of the developed PyFR SS-based operators with machine learning or optimization algorithms to permit automated decision support systems in dynamic and data-intensive environments;
- iii. Another promising direction includes extending the current approach to group DM settings by incorporating consensus mechanisms and conflict resolution strategies among multiple decision-makers under uncertainty.

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Conflicts of Interest

The author declares no conflicts of interest.

Appendix 1: Proof of Theorem 1

To prove the desired result, we use the well-known technique of mathematical induction on n as follows.

Step 1: For $n = 2$, we have:

$$= \left(\left(\left(\sqrt{1 - \left(\sum_{j=1}^2 \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^2 \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right), \right. \right. \\ \left. \left. \sqrt{\left(\sum_{j=1}^2 \mathbb{H}_j \left((\overline{\mathbb{E}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^2 \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \\ \left(\left(\left(\sqrt{1 - \left(\sum_{j=1}^2 \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^2 \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right), \right. \right. \\ \left. \left. \sqrt{\left(\sum_{j=1}^2 \mathbb{H}_j \left((\overline{\mathbb{E}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^2 \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \right)$$

thus, for $n = 2$, the result is true.

Step 2: Presume that Eq. (17) is valid for $n = \kappa$, then:

$$PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_\kappa) = \mathbb{H}_1 \overline{\mathcal{L}}_1 \oplus \mathbb{H}_2 \overline{\mathcal{L}}_2 \oplus \dots \oplus \mathbb{H}_\kappa \overline{\mathcal{L}}_\kappa = \bigoplus_{j=1}^\kappa \mathbb{H}_j \overline{\mathcal{L}}_j$$

$$= \left(\left(\left(\sqrt{1 - \left(\sum_{j=1}^\kappa \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^\kappa \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right), \right. \right. \\ \left. \left. \sqrt{\left(\sum_{j=1}^\kappa \mathbb{H}_j \left((\overline{\mathbb{E}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^\kappa \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \\ \left(\left(\left(\sqrt{1 - \left(\sum_{j=1}^\kappa \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^\kappa \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right), \right. \right. \\ \left. \left. \sqrt{\left(\sum_{j=1}^\kappa \mathbb{H}_j \left((\overline{\mathbb{E}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^\kappa \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \right)$$

Step 3: To demonstrate that Eq. (17) hold for $n = \kappa + 1$, consider:

$$PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_\kappa, \overline{\mathcal{L}}_{\kappa+1}) = \mathbb{H}_1 \overline{\mathcal{L}}_1 \oplus \mathbb{H}_2 \overline{\mathcal{L}}_2 \oplus \dots \oplus \mathbb{H}_\kappa \overline{\mathcal{L}}_\kappa \oplus \mathbb{H}_{\kappa+1} \overline{\mathcal{L}}_{\kappa+1} = \bigoplus_{j=1}^\kappa \mathbb{H}_j \overline{\mathcal{L}}_j \oplus \mathbb{H}_{\kappa+1} \overline{\mathcal{L}}_{\kappa+1}$$

$$= \left(\left(\left(\left(\sqrt{1 - \left(\sum_{j=1}^\kappa \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^\kappa \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right), \right. \right. \right. \\ \left. \left. \sqrt{\left(\sum_{j=1}^\kappa \mathbb{H}_j \left((\overline{\mathbb{E}}_j^L)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^\kappa \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \right) \oplus \left(\left(\left(\left(\sqrt{1 - \left(\mathbb{H}_{\kappa+1} \left(1 - (\overline{\mathbb{Q}}_{\kappa+1}^L)^2 \right)^{\overline{\vartheta}} - (\mathbb{H}_{\kappa+1} - 1) \right)^{\frac{1}{\overline{\vartheta}}}} \right), \right. \right. \right. \\ \left. \left. \sqrt{\left(\mathbb{H}_{\kappa+1} \left((\overline{\mathbb{E}}_{\kappa+1}^L)^2 \right)^{\overline{\vartheta}} - (\mathbb{H}_{\kappa+1} - 1) \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \right) \\ \left(\left(\left(\left(\sqrt{1 - \left(\sum_{j=1}^\kappa \mathbb{H}_j \left(1 - (\overline{\mathbb{Q}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^\kappa \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right), \right. \right. \right. \\ \left. \left. \sqrt{\left(\sum_{j=1}^\kappa \mathbb{H}_j \left((\overline{\mathbb{E}}_j^U)^2 \right)^{\overline{\vartheta}} - \sum_{j=1}^\kappa \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \right) \oplus \left(\left(\left(\left(\sqrt{1 - \left(\mathbb{H}_{\kappa+1} \left(1 - (\overline{\mathbb{Q}}_{\kappa+1}^U)^2 \right)^{\overline{\vartheta}} - (\mathbb{H}_{\kappa+1} - 1) \right)^{\frac{1}{\overline{\vartheta}}}} \right), \right. \right. \right. \\ \left. \left. \sqrt{\left(\mathbb{H}_{\kappa+1} \left((\overline{\mathbb{E}}_{\kappa+1}^U)^2 \right)^{\overline{\vartheta}} - (\mathbb{H}_{\kappa+1} - 1) \right)^{\frac{1}{\overline{\vartheta}}}} \right) \right) \right)$$

$$= \left(\begin{array}{c} \left(\sqrt{1 - \left(\sum_{j=1}^{\kappa+1} \mathbb{H}_j (1 - (\overline{\mathbb{Q}}_j^L)^2)^{\overline{\vartheta}} - \sum_{j=1}^{\kappa+1} \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right. \\ \left. \sqrt{\left(\sum_{j=1}^{\kappa+1} \mathbb{H}_j ((\overline{\mathbb{E}}_j^L)^2)^{\overline{\vartheta}} - \sum_{j=1}^{\kappa+1} \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \\ \left(\sqrt{1 - \left(\sum_{j=1}^{\kappa+1} \mathbb{H}_j (1 - (\overline{\mathbb{Q}}_j^U)^2)^{\overline{\vartheta}} - \sum_{j=1}^{\kappa+1} \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right. \\ \left. \sqrt{\left(\sum_{j=1}^{\kappa+1} \mathbb{H}_j ((\overline{\mathbb{E}}_j^U)^2)^{\overline{\vartheta}} - \sum_{j=1}^{\kappa+1} \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \end{array} \right).$$

Thus, the result is valid for $n = \kappa + 1$. Consequently, in the light of Step 1 to Step 3, it becomes apparent that Eq. (17) is true for all positive integers n .

Appendix 2: Proof of Theorem 2

Consider $\overline{\mathcal{L}} = \overline{\mathcal{L}}_j = \left((\overline{\mathbb{Q}}_j^L, \overline{\mathbb{E}}_j^L), (\overline{\mathbb{Q}}_j^U, \overline{\mathbb{E}}_j^U) \right), \forall j$. Then, by using Eq. (17), we have:

$$\begin{aligned} \text{PyFRSSPA}(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) &= \left(\begin{array}{c} \left(\sqrt{1 - \left(\sum_{j=1}^n \mathbb{H}_j (1 - (\overline{\mathbb{Q}}^L)^2)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right. \\ \left. \sqrt{\left(\sum_{j=1}^n \mathbb{H}_j ((\overline{\mathbb{E}}^L)^2)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \\ \left(\sqrt{1 - \left(\sum_{j=1}^n \mathbb{H}_j (1 - (\overline{\mathbb{Q}}^U)^2)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right. \\ \left. \sqrt{\left(\sum_{j=1}^n \mathbb{H}_j ((\overline{\mathbb{E}}^U)^2)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \end{array} \right) \\ &= \left(\left(\sqrt{1 - \left((1 - (\overline{\mathbb{Q}}^L)^2)^{\overline{\vartheta}} \right)^{\frac{1}{\overline{\vartheta}}}} \right), \sqrt{\left((\overline{\mathbb{E}}^L)^2 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \left(\sqrt{1 - \left((1 - (\overline{\mathbb{Q}}^U)^2)^{\overline{\vartheta}} \right)^{\frac{1}{\overline{\vartheta}}}} \right), \sqrt{\left((\overline{\mathbb{E}}^U)^2 \right)^{\frac{1}{\overline{\vartheta}}}} \right) \\ &= \left(\left(\sqrt{1 - (1 - (\overline{\mathbb{Q}}^L)^2)}, \sqrt{(\overline{\mathbb{E}}^L)^2} \right) \left(\sqrt{1 - (1 - (\overline{\mathbb{Q}}^U)^2)}, \sqrt{(\overline{\mathbb{E}}^U)^2} \right) \right) = ((\overline{\mathbb{Q}}^L, \overline{\mathbb{E}}^L), (\overline{\mathbb{Q}}^U, \overline{\mathbb{E}}^U)) = \overline{\mathcal{L}}. \end{aligned}$$

Appendix 3: Proof of Theorem 3

Consider if $\overline{\mathcal{L}}_j \leq \overline{\mathcal{L}}'_j$, then $\overline{\mathbb{Q}}_j^L \leq \overline{\mathbb{Q}}'^L_j, \overline{\mathbb{E}}^L \geq \overline{\mathbb{E}}'^L, \overline{\mathbb{Q}}_j^U \leq \overline{\mathbb{Q}}'^U_j$ and $\overline{\mathbb{E}}^U \geq \overline{\mathbb{E}}'^U$. Now, consider:

$$\begin{aligned} \overline{\mathbb{Q}}_j^L \leq \overline{\mathbb{Q}}'^L_j &\Rightarrow 1 - \overline{\mathbb{Q}}_j^L \geq 1 - \overline{\mathbb{Q}}'^L_j \\ \Rightarrow (1 - \overline{\mathbb{Q}}_j^L)^{\overline{\vartheta}} &\leq (1 - \overline{\mathbb{Q}}'^L_j)^{\overline{\vartheta}} \\ \Rightarrow \sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^L)^{\overline{\vartheta}} &\leq \sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}'^L_j)^{\overline{\vartheta}} \\ \Rightarrow \sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^L)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 &\leq \sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}'^L_j)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \geq \sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^U)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \end{aligned}$$

$$\Rightarrow \left(\sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^L)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}} \geq \left(\sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^L)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}} \geq \left(\sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^L)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}$$

$$\Rightarrow \left(\sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^L)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}} \leq \left(\sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^L)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}} \leq \left(\sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^L)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}.$$

Further, we consider:

$$\overline{\mathbb{E}}_j^{\overline{U}} \geq \overline{\mathbb{E}}_j^{\prime\overline{U}} \Rightarrow \sum_{j=1}^n \mathbb{H}_j \overline{\mathbb{E}}_j^{\overline{U}} \geq \sum_{j=1}^n \mathbb{H}_j \overline{\mathbb{E}}_j^{\prime\overline{U}}$$

$$\Rightarrow \sum_{j=1}^n \mathbb{H}_j \overline{\mathbb{E}}_j^{\overline{U}} - \sum_{j=1}^n \mathbb{H}_j + 1 \geq \sum_{j=1}^n \mathbb{H}_j \overline{\mathbb{E}}_j^{\prime\overline{U}} - \sum_{j=1}^n \mathbb{H}_j + 1$$

$$\Rightarrow \left(\sum_{j=1}^n \mathbb{H}_j \overline{\mathbb{E}}_j^{\overline{U}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}} \geq \left(\sum_{j=1}^n \mathbb{H}_j \overline{\mathbb{E}}_j^{\prime\overline{U}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}.$$

Similarly, we can show that:

$$\left(\sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^U)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}} \geq \left(\sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^U)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}} \geq \left(\sum_{j=1}^n \mathbb{H}_j (1 - \overline{\mathbb{Q}}_j^U)^{\overline{\vartheta}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}$$

and

$$\left(\sum_{j=1}^n \mathbb{H}_j \overline{\mathbb{E}}_j^{\overline{L}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}} \geq \left(\sum_{j=1}^n \mathbb{H}_j \overline{\mathbb{E}}_j^{\prime\overline{L}} - \sum_{j=1}^n \mathbb{H}_j + 1 \right)^{\frac{1}{\overline{\vartheta}}}.$$

Consequently, it follows that $PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq PyFRSSPA(\overline{\mathcal{L}}_1^+, \overline{\mathcal{L}}_2^+, \dots, \overline{\mathcal{L}}_n^+)$.

Appendix 4: Proof of Theorem 4

According to Theorem 2 and Theorem 3, we have:

$$PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \geq PyFRSSPA(\overline{\mathcal{L}}_1^-, \overline{\mathcal{L}}_2^-, \dots, \overline{\mathcal{L}}_n^-) = \overline{\mathcal{L}}_j^-$$

and

$$PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq PyFRSSPA(\overline{\mathcal{L}}_1^+, \overline{\mathcal{L}}_2^+, \dots, \overline{\mathcal{L}}_n^+) = \overline{\mathcal{L}}_j^+.$$

Therefore:

$$\overline{\mathcal{L}}_j^- \leq PyFRSSPA(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \dots, \overline{\mathcal{L}}_n) \leq \overline{\mathcal{L}}_j^+.$$

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