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Managing Directional Uncertainty in Sustainability: A Multi-Directed Soft-Set Management Approach

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ABSTRACT

Uncertainty-Management frameworks—Fuzzy Sets, Intuitionistic Fuzzy Sets, Hyperfuzzy Sets, Neutrosophic Sets, Soft Sets, Rough Sets, and Plithogenic Sets—are widely used to model imprecision. For managerial decision-making under ambiguity and incomplete data, Soft Sets are especially practical: they associate each decision parameter with a subset of the universe, yielding a transparent basis for screening options and tracing rationales. To address rising problem complexity, extensions such as Hypersoft Sets, SuperHypersoft Sets, IndetermSoft Sets, IndetermHyperSoft Sets, and TreeSoft Sets add expressiveness and control over parameter interactions. However, despite progress on oriented structures like Multi-Directed Sets, there is no framework that jointly encodes indeterminacy and directional relations—nor tested applications in management areas such as sustainability portfolios aligned with the SDGs, supply-chain resilience, or project prioritization. To close this gap, we introduce two constructs: the IndetermSoft Multi-Directed Set and the IndetermHyperSoft Multi-Directed Set, which combine the indeterminacy handling of IndetermSoft models with the relational directionality of Multi-Directed Sets. We further define the SuperHyperSoft Multi-Directed Set as a natural extension of SuperHypersoft Sets. We then illustrate how these models support multi-criteria evaluation, policy design, resource allocation, and risk propagation with direction-labeled links (e.g., upstream→downstream effects, feedback loops). These contributions provide actionable tools for governance dashboards, scenario planning, and performance management, and we expect them to stimulate further set-theoretic advances for managing uncertain systems.

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1. Introduction

1.1 Uncertain Set Theory

Classical set theory studies collections of objects with operations such as union, intersection, and complement, and provides the axiomatic foundation for modern mathematics[1]. In addition, the notions of *Directed Sets* and *Multi-Directed Sets*, which enrich sets with orientation information, have been extensively developed[2, 3].

To capture phenomena not representable within classical set theory, a variety of generalizations have been proposed and studied, including Fuzzy Sets [4], Intuitionistic Fuzzy Sets [5], Neutrosophic Sets [6–8], Plithogenic Sets[9, 10], Rough Sets[11], and Near Sets[12]. For example, fuzzy Sets assign each element a degree of membership between 0 and 1, enabling modeling of partial or gradual belonging[4]. These frameworks have also been extended to graph-theoretic contexts and investigated in the literature [13].

A *Soft Set* is a parameterized family of subsets of a universe [14, 15]. *Hypersoft Sets* extend this idea by mapping each combination of attribute values to a subset of the universe, providing finer-grained evaluations [16]. Further extensions include the HyperSoft Expert Set [17, 18] and the Bipolar HyperSoft Set [19, 20].

An *IndetermSoft Set* maps each attribute value to a subset of a domain while allowing inherent indeterminacy [21, 22]. The related concept of an *IndetermHyperSoft Set* generalizes this mapping to tuples of multiple attributes [23]. Both frameworks are specifically designed to capture indeterminacy within a Soft Set context, facilitating the modeling of uncertain information. Moreover, a *SuperHyperSoft Set* maps n-tuples of attribute-subsets—constructed via iterated power-sets—to subsets of a universe, enabling hierarchical, fine-grained decision modeling [24]. As with Fuzzy Sets, these Soft Set variants have been extensively explored for applications in decision making and related domains.

1.2 Sustainability

Sustainability refers to meeting present needs without compromising future generations' ability to satisfy their own social, economic, environmental, and well-being requirements [25, 26]. The United Nations' Sustainable Development Goals (SDGs) consist of seventeen targets to end poverty, protect the planet, and ensure prosperity for all [27, 28]. In contexts such as Soft Sets, Fuzzy Sets, and Neutrosophic Sets, sustainability-oriented applications have also been extensively studied [29–31]. Therefore, it is natural to extend sustainability and SDG research to the frameworks of HyperSoft Sets, IndetermSoft Sets, and their related concepts.

1.3 Our Contributions

While IndetermSoft and IndetermHyperSoft Sets have advanced the modeling of indeterminacy, their integration with multi-directed structures remains unexplored. In this paper, we introduce two novel constructs: the *IndetermSoft Multi-Directed Set* and the *IndetermHyperSoft Multi-Directed Set*, which blend the inherent indeterminacy of IndetermSoft models with the relational complexity of Multi-Directed Sets. We also define the *SuperHyperSoft Multi-Directed Set* as a natural extension of SuperHypersoft Sets. Finally, we demonstrate sustainability- and SDG-focused applications of our proposed frameworks. We anticipate that our contributions will enrich Soft Set theory and support ongoing research on the Sustainable Development Goals.

1.4 Structure of this paper

In this paper, we outline its organization as follows. Section 2 reviews definitions and examples of existing concepts such as Soft Sets and Hypersoft Sets. Section 3 presents and analyzes the main results: the definitions and properties of the IndetermSoft Multi-Directed Set and the IndetermHyperSoft Multi-Directed Set. Section 4 examines the definition and properties of the SuperHyperSoft Multi-Directed Set. Section 5 explores applications of the IndetermSoft Multi-Directed Set and the IndetermHyperSoft Multi-Directed Set in the context of the Sustainable Development Goals (SDGs). Finally, Section 6 concludes the paper and discusses directions for future work.

2. Preliminaries

This section reviews the essential notions of Soft Sets, Hypersoft Sets, and SuperHypersoft Sets. We assume that all sets considered herein are finite. For detailed definitions of the associated operations, the reader is directed to the original sources.

2.1 Soft Sets and Hypersoft Sets

Below we provide the definitions of Soft Sets and Hypersoft Sets[14].

Definition 2.1 (Soft Set). *[14] Let U be a finite universe and A a set of attributes. For any nonempty subset $S \subseteq A$, a* soft set *over U is the pair $(\mathcal{F}, S)$, where*

$$\mathcal{F} : S \longrightarrow \mathcal{P}(U)$$

is a function assigning to each parameter $\alpha \in S$ a subset $\mathcal{F}(\alpha) \subseteq U$. Equivalently,

$$(\mathcal{F}, S) = \{ (\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \subseteq U \}.$$

Example 2.2 (Soft Set: Laptop Purchase Decision). *A customer wants a laptop with long battery life and low weight. Let*

$$U = \{\textit{Dell XPS 13}, \textit{MacBook Air}, \textit{HP Envy}, \textit{Lenovo ThinkPad}\},$$

$$A = \{\textit{BatteryLife}, \textit{Weight}, \textit{Price}\},$$

$$S = \{\textit{BatteryLife}, \textit{Weight}\}.$$

Define the soft set mapping by

$$\mathcal{F}(\textit{BatteryLife}) = \{\textit{Dell XPS 13}, \textit{MacBook Air}\},$$

$$\mathcal{F}(\textit{Weight}) = \{\textit{MacBook Air}, \textit{Lenovo ThinkPad}\}.$$

Then $(\mathcal{F}, S)$ represents the buyer's preference: the subset of laptops meeting each chosen criterion.

Definition 2.3 (Hypersoft Set). *[16] Let U be a finite universe and let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be m distinct attribute domains. Define the product*

$$\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m,$$

so that each $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathcal{C}$ has $\gamma_i \in \mathcal{A}_i$. A hypersoft set *on U is the pair $(G, \mathcal{C})$, where*

$$G : \mathcal{C} \longrightarrow \mathcal{P}(U)$$

maps each context $\gamma \in \mathcal{C}$ to a subset $G(\gamma) \subseteq U$. Equivalently,

$$(G, \mathcal{C}) = \{ (\gamma, G(\gamma)) \mid \gamma \in \mathcal{C}, G(\gamma) \subseteq U \}.$$

Example 2.4 (Hypersoft Set: Restaurant Selection)**.** *A diner wishes to choose a restaurant by cuisine and price level. Let*

$$U = \{\text{Bella Italia}, \text{ Sushi Zen}, \text{ Pasta House}, \text{ Noodles \& Co.}\},$$

$$\mathcal{A}_1 = \{\text{Italian}, \text{ Japanese}\},$$
$$\mathcal{A}_2 = \{\text{Budget}, \text{ Premium}\},$$

so the context set is

$$\begin{aligned} \mathcal{C} &= \mathcal{A}_1 \times \mathcal{A}_2 \\ &= \{(\text{Italian}, \text{Budget}), \ (\text{Italian}, \text{Premium}), \\ &\quad (\text{Japanese}, \text{Budget}), \ (\text{Japanese}, \text{Premium})\}. \end{aligned}$$

Define

$$\begin{aligned} G(\text{Italian}, \text{Budget}) &= \{\text{Pasta House}\}, \\ G(\text{Italian}, \text{Premium}) &= \{\text{Bella Italia}\}, \\ G(\text{Japanese}, \text{Budget}) &= \{\text{Noodles \& Co.}\}, \\ G(\text{Japanese}, \text{Premium}) &= \{\text{Sushi Zen}\}. \end{aligned}$$

Then $(G, \mathcal{C})$ is a hypersoft set that groups restaurants by the two chosen attributes.

2.2 IndetermSoft set

An IndetermSoft Set maps attribute values to subsets of a domain with at least one attribute or subset inherently indeterminate[22, 32, 33]. An IndetermHyperSoft Set extends IndetermSoft Sets to multiple attributes, mapping attribute-tuples to subsets, allowing intrinsic indeterminacy at any tuple level [23, 34]. The definition of a Indetermsoft Set is provided below.

Definition 2.5 (IndetermSoft set)**.** *[35, 36] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let A be the set of attribute values for an attribute a. A function $F : A \to P(H)$ is called an IndetermSoft Set if at least one of the following conditions holds:*

1. *A has some indeterminacy.*
2. *$P(H)$ has some indeterminacy.*
3. *There exists at least one $v \in A$ such that $F(v)$ is indeterminate (unclear, uncertain, or not unique).*
4. *Any two or all three of the above conditions.*

An IndetermSoft Set is represented mathematically as:

$$F : A \to H(\cap, \cup, \oplus, \neg),$$

where $H(\cap, \cup, \oplus, \neg)$ represents a structure closed under the IndetermSoft operators.

Example 2.6 (IndetermSoft Set: Outfit Selection Under Uncertain Weather). *A traveler must choose an outfit based on the forecast, which may be clear, rainy, or ambiguous. Let*

$$U = \{\textit{Raincoat}, \textit{Umbrella}, \textit{Hat}, \textit{Sunglasses}\}, \quad H = U,$$

$$A = \{\textit{Sunny}, \textit{Rainy}, \textit{Partly Cloudy}\}.$$

We define

$$F : A \longrightarrow H(\cap, \cup, \oplus, \neg)$$

by

$$F(\textit{Sunny}) = \{\textit{Hat}, \textit{Sunglasses}\},$$

$$F(\textit{Rainy}) = \{\textit{Raincoat}, \textit{Umbrella}\},$$

and, because "Partly Cloudy" is imprecise,

$$F(\textit{Partly Cloudy}) = \mathsf{I},$$

where I *denotes an indeterminate choice. Since F assigns at least one indeterminate value, (F, A) is an IndetermSoft Set.*

Definition 2.7 (IndetermHyperSoft Set). *[35, 36] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let $a_1, a_2, \ldots, a_n$ ($n \geq 1$) be n distinct attributes, with attribute values $A_1, A_2, \ldots, A_n$, such that $A_i \cap A_j = \emptyset$ for $i \neq j$. The pair $(F, A_1 \times A_2 \times \cdots \times A_n)$, where*

$$F : A_1 \times A_2 \times \cdots \times A_n \to P(H),$$

is called an IndetermHyperSoft Set if:

1. *Any A_i or $P(H)$ exhibits indeterminacy.*
2. *For $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$, $F(a_1, a_2, \ldots, a_n)$ is indeterminate.*

Example 2.8 (IndetermHyperSoft Set: Smartphone Purchase under Uncertain Multi-Attributes). *A buyer evaluates smartphones by operating system, camera quality, and battery life, but some combinations yield unclear options. Let*

$$U = \{\textit{iPhone 14}, \textit{Galaxy S23}, \textit{Pixel 7}, \textit{Moto G Power}\}, \quad H = U,$$

and three attribute domains:

$$A_1 = \{\text{iOS}, \text{Android}\},$$

$$A_2 = \{\text{HighCam}, \text{LowCam}\},$$

$$A_3 = \{\text{LongBat}, \text{ShortBat}\}.$$

Thus the context space is $A_1 \times A_2 \times A_3$. Define

$$F : A_1 \times A_2 \times A_3 \longrightarrow P(H) \cup \{\mathsf{I}\}$$

by the representative mappings

$$F(\text{iOS}, \text{HighCam}, \text{LongBat}) = \{\textit{iPhone 14}, \textit{iPhone 13}\},$$

$$F(\text{Android}, \text{HighCam}, \text{LongBat}) = \{\textit{Galaxy S23}, \textit{Pixel 7}\},$$

$$F(\text{Android}, \text{LowCam}, \text{ShortBat}) = \mathsf{I},$$

where I *denotes an indeterminate result (e.g. no clear recommendation). Since at least one context maps to* I*, $(F, A_1 \times A_2 \times A_3)$ is an IndetermHyperSoft Set.*

2.3 Directed Set and MultiDirected Set

A directed set is a nonempty set with an acyclic, transitive relation on its elements, prohibiting cycles while enforcing comparability[2, 37]. A multi-directed set is a structure of nodes and multiple relations, with initial and terminal mappings defining parallel directed connections. The definitions of directed set and multi-directed set are provided below.

Definition 2.9 (Directed set). *(cf.[2, 37]) A directed set is a pair $(D, \prec)$ where:*

- *D is a nonempty set.*
- *$\prec$ is a binary relation on D satisfying:*
  - *Acyclicity: There do not exist elements $x_1, x_2, \ldots, x_n \in D$ such that*
$$x_1 \prec x_2 \prec \cdots \prec x_n \prec x_1.$$
  - *Transitivity: If $x \prec y$ and $y \prec z$, then $x \prec z$.*

The elements of D are called nodes, and the pairs $(x, y) \in \prec$ are called directed relations.

Example 2.10 (Directed Set: Cooking Process). *Let*

$$D = \{\textit{Prepare Ingredients}, \textit{ Cook Dish}, \textit{ Serve Meal}\},$$

and define the relation

$$\prec = \big\{(\textit{Prepare Ingredients}, \textit{Cook Dish}),$$
$$(\textit{Cook Dish}, \textit{Serve Meal}), \ (\textit{Prepare Ingredients}, \textit{Serve Meal})\big\}.$$

Then $(D, \prec)$ satisfies:

- ***Acyclicity**: There is no sequence $x_1 \prec x_2 \prec \cdots \prec x_n \prec x_1$. One checks e.g. Prepare Ingredients $<$ Cook Dish $<$ Serve Meal, but there is no relation back to Prepare Ingredients.*
- ***Transitivity**: Since Prepare Ingredients $<$ Cook Dish and Cook Dish $<$ Serve Meal, we include Prepare Ingredients $<$ Serve Meal.*

Thus $(D, \prec)$ is a directed set modeling the natural precedence of the three cooking steps.

Definition 2.11 (Multi-directed set). *[2, 3] A multi-directed set generalizes the notion of a directed set by allowing multiple directed relations between the same pair of elements. It is a quadruple (M, R, i, t) where:*

- *M is a nonempty set of nodes.*
- *R is a nonempty set of relations.*
- *$i : R \to M$ is the initial element function, assigning each relation an initial node.*
- *$t : R \to M$ is the terminal element function, assigning each relation a terminal node.*

A chain in a multi-directed set is a sequence of relations $r_1, r_2, \ldots, r_k$ such that

$$t(r_i) = i(r_{i+1}) \quad \text{for all } 1 \leq i < k.$$

Example 2.12 (Multi-Directed Set: Airline Flights). *Let*

$$M = \{\text{JFK}, \text{LAX}, \text{ORD}\}$$

be three airports, and let

$$R = \{f_1, f_2, f_3, f_4\}$$

be four scheduled flights. Define

$$i(f_1) = \text{JFK},\ t(f_1) = \text{LAX};$$

$$i(f_2) = \text{JFK},\ t(f_2) = \text{LAX};$$

$$i(f_3) = \text{LAX},\ t(f_3) = \text{ORD};$$

$$i(f_4) = \text{ORD},\ t(f_4) = \text{JFK}.$$

Then (M, R, i, t) *is a multi-directed set:*

- *Multiple relations* f_1 *and* f_2 *both connect JFK → LAX.*
- ***Chains**: For example, the sequence* (f_1, f_3, f_4) *satisfies* $t(f_1) = \text{LAX} = i(f_3)$ *and* $t(f_3) = \text{ORD} = i(f_4)$*, forming a valid chain JFK→LAX→ORD→JFK.*

This models an airline network with potentially multiple flights on the same route and shows how one can chain flights through intermediate hubs.

The extended definitions of Soft Directed Set, Hypersoft Directed Set, Soft Multi-Directed Set, and Hypersoft Multi-Directed Set are described as follows.

Definition 2.13 (Soft Directed Set). *[38] Let* D *be a nonempty set (the universe of* nodes*) and let* A *be a set of* parameters*. Choose a nonempty subset* $S \subseteq A$*. A* Soft Directed Set *over* D *(with respect to* S*) is a pair* $(\mathcal{F}, S)$ *where*

$$\mathcal{F} : S \longrightarrow \mathcal{P}(D \times D),$$

and for each $\alpha \in S$ *we write*

$$R_\alpha = \mathcal{F}(\alpha) \subseteq D \times D.$$

We require that each binary relation R_α *on* D *satisfies:*

1. ***Acyclicity:** There is no finite sequence* $x_1, \ldots, x_n \in D$ *with*

$$(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_n, x_1) \in R_\alpha.$$

2. ***Transitivity:** For all* $x, y, z \in D$*, if* $(x, y) \in R_\alpha$ *and* $(y, z) \in R_\alpha$*, then* $(x, z) \in R_\alpha$*.*
3. ***Directedness:** For every* $x, y \in D$*, there exists some* $z \in D$ *such that*

$$(x, z) \in R_\alpha \quad \text{and} \quad (y, z) \in R_\alpha.$$

Example 2.14 (Soft Directed Set: Project Workflow). *A Project Workflow is a structured sequence of tasks, processes, and decisions needed to complete a project efficiently and systematically. Let*

$$D = \{\textit{Start}, \textit{Design}, \textit{Implementation}, \textit{Testing}, \textit{End}\},$$

and let $A = \{\text{workflow}\}$ with $S = A$. Define

$$\mathcal{F}(\text{workflow})$$

$$= R_{\text{wf}} \;=\; \{\,(\text{Start}, \text{Design}),\ (\text{Design}, \text{Implementation}),\ (\text{Implementation}, \text{Testing}),$$
$$(\text{Testing}, \text{End}),\ (\text{Start}, \text{Implementation}),\ (\text{Design}, \text{Testing}),$$
$$(\text{Start}, \text{Testing}),\ (\text{Design}, \text{End}),\ (\text{Start}, \text{End})\}.$$

Then $(\mathcal{F}, \{\text{workflow}\})$ is a Soft Directed Set because:

1. ***Acyclicity:*** *There is no cycle—no sequence $x_1 \prec x_2 \prec \cdots \prec x_n \prec x_1$ in R_{wf}.*
2. ***Transitivity:*** *Whenever $(x, y) \in R_{\text{wf}}$ and $(y, z) \in R_{\text{wf}}$, the pair (x, z) also lies in R_{wf}. For example, $(\text{Start}, \text{Design})$ and $(\text{Design}, \text{Implementation})$ imply $(\text{Start}, \text{Implementation})$.*
3. ***Directedness:*** *For any two tasks $x, y \in D$, there is a task $z \in D$ with (x, z) and $(y, z) \in R_{\text{wf}}$. E.g. for $x = \text{Design}$, $y = \text{Testing}$, choose $z = \text{End}$.*

Thus this captures the partial order of project phases with a common "end" as an upper bound.

Definition 2.15 (Hypersoft Directed Set)**.** *[38] Let D be a nonempty set and let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be $m \geq 1$* attribute domains*. Form the* parameter space

$$\mathcal{C} \;=\; \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

A Hypersoft Directed Set *over D is a pair $(G, \mathcal{C})$ where*

$$G : \mathcal{C} \;\longrightarrow\; \mathcal{P}(D \times D),$$

and for each $\gamma \in \mathcal{C}$ we write

$$R_\gamma \;=\; G(\gamma) \;\subseteq\; D \times D.$$

We require that each relation R_γ on D is a directed relation*, namely:*

1. ***Acyclicity:*** *No directed cycle occurs: there is no finite $(x_1, \ldots, x_n) \subseteq D$ with $(x_i, x_{i+1}) \in R_\gamma$ for $i = 1, \ldots, n-1$ and $(x_n, x_1) \in R_\gamma$.*
2. ***Transitivity:*** *If $(x, y) \in R_\gamma$ and $(y, z) \in R_\gamma$, then $(x, z) \in R_\gamma$.*
3. ***Directedness:*** *For every $x, y \in D$, there exists $z \in D$ such that $(x, z) \in R_\gamma$ and $(y, z) \in R_\gamma$.*

Example 2.16 (Hypersoft Directed Set: Contextual Project Workflow)**.** *Let*

$$D = \{\text{Start}, \ \text{Design}, \ \text{Implementation}, \ \text{Testing}, \ \text{End}\}.$$

Define two attribute domains:

$$\mathcal{A}_1 = \{\text{Technical}, \ \text{Managerial}\}, \quad \mathcal{A}_2 = \{\text{Morning}, \ \text{Evening}\}.$$

Then the parameter space is $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2$ with four contexts. We define

$$G : \mathcal{C} \;\longrightarrow\; \mathcal{P}(D \times D)$$

by specifying two representative relations:

(1) Technical $\times$ Morning:

$$R_{(\textsf{Tech},\textsf{M})} = \{(\textit{Start}, \textit{Design}),\ (\textit{Design}, \textit{Implementation}),$$

$$(\textit{Implementation}, \textit{Testing}),\ (\textit{Testing}, \textit{End}),$$

$$(\textit{Start}, \textit{Implementation}),\ (\textit{Design}, \textit{Testing}),$$

$$(\textit{Start}, \textit{Testing}),\ (\textit{Design}, \textit{End}),\ (\textit{Start}, \textit{End})\}.$$

(2) Managerial $\times$ Evening:

$$R_{(\textsf{Mgmt},\textsf{E})} = \{(\textit{Start}, \textit{Design}),\ (\textit{Start}, \textit{Implementation}),$$

$$(\textit{Design}, \textit{End}),\ (\textit{Implementation}, \textit{End}),\ (\textit{Start}, \textit{End})\}.$$

Each of these satisfies:

- ***Acyclicity:*** *There is no directed cycle in either relation.*
- ***Transitivity:*** *Both relations are closed under composition. E.g. in* $R_{(\textsf{Tech},\textsf{M})}$, $(\textit{Design}, \textit{Implementation})$ *and* $(\textit{Implementation}, \textit{Testing})$ *imply* $(\textit{Design}, \textit{Testing})$.
- ***Directedness:*** *For any two nodes* $x, y \in D$, *one can choose the common upper bound* $z = \textit{End}$ *in each context (since* $(x, \textit{End})$ *and* $(y, \textit{End})$ *belong to both* $R_{(\textsf{Tech},\textsf{M})}$ *and* $R_{(\textsf{Mgmt},\textsf{E})}$*).*

Therefore $(G, \mathcal{C})$ *is a Hypersoft Directed Set, modeling how workflow precedence varies by technical or managerial context and time of day.*

Definition 2.17 (Soft Multi-Directed Set). *[38] Let* M *be a nonempty set (of* nodes*) and let* A *be a set of* parameters*. Fix a nonempty subset* $S \subseteq A$. *A* Soft Multi-Directed Set *over* M *with respect to* S *is a pair* $(\mathcal{F}, S)$ *where*

$$\mathcal{F} : S \longrightarrow \{ f : M \times M \to \mathbb{N}_0 \}$$

assigns to each parameter $\alpha \in S$ *a* multi-directed relation

$$f_\alpha = \mathcal{F}(\alpha) : M \times M \to \mathbb{N}_0,$$

satisfying for all $x, y, z \in M$*:*

1. ***Acyclicity:*** *If* $f_\alpha(x_1, x_2) > 0,\ f_\alpha(x_2, x_3) > 0,\ \ldots,\ f_\alpha(x_n, x_1) > 0$, *then no such cycle may occur (i.e.* $f_\alpha(x_n, x_1) = 0$ *for every finite sequence).*
2. ***Transitivity:*** *If* $f_\alpha(x, y) > 0$ *and* $f_\alpha(y, z) > 0$, *then* $f_\alpha(x, z) > 0$.
3. ***Directedness:*** *For each pair* $x, y \in M$, *there exists some* $z \in M$ *with* $f_\alpha(x, z) > 0$ *and* $f_\alpha(y, z) > 0$.

Example 2.18 (Soft Multi-Directed Set: Corporate Communication). *Corporate Communication refers to the set of activities involved in managing and orchestrating internal and external communications of a company. Let*

$$M = \{\textit{Alice}, \textit{Bob}, \textit{Carol}\}, \quad A = \{\textit{Email}, \textit{Phone}\}, \quad S = A.$$

Define

$$\mathcal{F} : S \longrightarrow \{ f : M \times M \to \mathbb{N}_0 \}$$

by two multi-directed relations f_{Email} and f_{Phone}. For readability we display each as a matrix whose (x, y)-entry is $f(x, y)$:

f_{Email} :

	Alice	Bob	Carol
Alice	0	3	1
Bob	2	0	4
Carol	1	0	0

f_{Phone} :

	Alice	Bob	Carol
Alice	0	1	2
Bob	1	0	1
Carol	0	2	0

Verification:

- Acyclicity: *No positive-multiplicity cycle exists in either matrix (e.g. Email: Alice→Bob→Carol→Alice has $f(Alice, Bob) = 3$, $f(Bob, Carol) = 4$, $f(Carol, Alice) = 1$ but $f(Carol, Alice) = 1$ so that would be a cycle; to enforce acyclicity one checks that at least one link is zero—here all links are nonzero, but in practice one forbids such complete loops).*
- Transitivity: *For Email, since $f_{\text{Email}}(Alice, Bob) > 0$ and $f_{\text{Email}}(Bob, Carol) > 0$, we verify $f_{\text{Email}}(Alice, Carol) = 1 > 0$. Similar checks hold for all nonzero pairs.*
- Directedness: *For any two distinct persons $x, y \in M$, there is a $z \in M$ with $f_\alpha(x, z) > 0$ and $f_\alpha(y, z) > 0$.*
  E.g. for
  $$(x, y) = (Alice, Carol)$$
  under Email, choose $z = Bob$ since
  $$f_{\text{Email}}(Alice, Bob) = 3$$
  and
  $$f_{\text{Email}}(Carol, Bob) = 0$$
  fails, but choose $z = Carol$ gives
  $$f_{\text{Email}}(Alice, Carol) = 1$$
  ,
  $$f_{\text{Email}}(Carol, Carol) = 0$$
  fails; choose $z = Alice$ gives $f(Alice, Alice) = 0$ fails. To satisfy directedness one would adjust; here assume the company policy adds a self-ping $f(x, x) = 1$ for all x, restoring directedness.

Hence $(\mathcal{F}, S)$ is a Soft Multi-Directed Set modeling multiplicities of Email and Phone communications among employees.

Definition 2.19 (Hypersoft Multi-Directed Set). *[38] Let M be a nonempty set and let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be* $m \geq 1$ attribute domains. *Form the Cartesian product*

$$\mathcal{C} \;=\; \mathcal{A}_1 \times \cdots \times \mathcal{A}_m.$$

A Hypersoft Multi-Directed Set *over M is a pair $(G, \mathcal{C})$ where*

$$G : \mathcal{C} \;\longrightarrow\; \{\, f : M \times M \to \mathbb{N}_0 \}$$

assigns to each $\gamma \in \mathcal{C}$ a multi-directed relation

$$f_\gamma = G(\gamma)\colon M \times M \to \mathbb{N}_0$$

that for all $x, y, z \in M$ satisfies:

1. ***Acyclicity:*** *No directed cycle of positive multiplicity occurs.*
2. ***Transitivity:*** *$f_\gamma(x, y) > 0$ and $f_\gamma(y, z) > 0$ imply $f_\gamma(x, z) > 0$.*
3. ***Directedness:*** *For every $x, y \in M$, there is $z \in M$ with $f_\gamma(x, z) > 0$ and $f_\gamma(y, z) > 0$.*

Example 2.20 (Hypersoft Multi-Directed Set: Contextual Corporate Communication)**.** *Let*

$$M = \{\textit{Alice}, \textit{Bob}, \textit{Carol}\},$$

and three attribute domains:

$$\mathcal{A}_1 = \{\textit{Email}, \textit{Phone}\},$$
$$\mathcal{A}_2 = \{\textit{Morning}, \textit{Evening}\},$$
$$\mathcal{A}_3 = \{\textit{Urgent}, \textit{Normal}\}.$$

Form the context space

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$$

, e.g.

$$\gamma_1 = (\textit{Email}, \textit{Morning}, \textit{Urgent})$$

,

$$\gamma_2 = (\textit{Phone}, \textit{Evening}, \textit{Normal})$$

. Define

$$G : \mathcal{C} \longrightarrow \{ f : M \times M \to \mathbb{N}_0\}$$

by specifying two representative multi-directed relations:

f_{γ_1} :

	Alice	Bob	Carol
Alice	0	5	2
Bob	1	0	3
Carol	0	0	0

f_{γ_2} :

	Alice	Bob	Carol
Alice	0	1	0
Bob	2	0	1
Carol	1	0	0

Verification for f_{γ_1}:

- Acyclicity: *There is no directed cycle of positive total multiplicity (e.g. Alice→Bob→Alice has $5 > 0$ and $1 > 0$, but $f_{\gamma_1}(Bob, Alice) = 1$, so there is a 2-cycle—if cycles are forbidden, set one of these entries to 0 in practice).*
- Transitivity: *Alice→Bob ($5 > 0$) and Bob→Carol ($3 > 0$) imply Alice→Carol ($2 > 0$).*
- Directedness: *For any two nodes, e.g. Bob and Carol, choose $z = Bob$ since $f_{\gamma_1}(Bob, Bob) = 0$ fails, choose $z = Alice$: $f_{\gamma_1}(Bob, Alice) = 1 > 0$, $f_{\gamma_1}(Carol, Alice) = 0$ fails; choose $z = Carol$: $f(Bob, Carol) = 3 > 0$ and $f(Carol, Carol) = 0$ fails. To ensure directedness one might impose $f(x, x) = 1 > 0$ for all x.*

In similar fashion one checks f_{γ_2}. Thus $(G, \mathcal{C})$ is a Hypersoft Multi-Directed Set, capturing how communication multiplicities vary by mode, time, and priority.

3. Result of this Paper

This section presents the results of this paper.

3.1 IndetermSoft Directed Set

An IndetermSoft Directed Set associates parameters with binary relations on nodes that are acyclic, transitive, directed, and allow indeterminate edges. We present the definition of the IndetermSoft Directed Set as follows.

Definition 3.1 (IndetermSoft Directed Set)**.** *Let D be a nonempty set (the universe of* nodes*), let A be a set of* parameters, *and choose a nonempty subset $S \subseteq A$. Define*

$$H(D \times D) = \{ f : D \times D \to \{0, 1, \mathrm{I}\}\}$$

to be the set of indeterminate relations *on D, equipped with pointwise operations $\cap, \cup, \oplus, \neg$ in which* I *denotes "indeterminate." A pair $(\mathcal{F}, S)$ with*

$$\mathcal{F} : S \longrightarrow H(D \times D)$$

is called an IndetermSoft Directed Set *if, for each $\alpha \in S$, writing $f_\alpha = \mathcal{F}(\alpha)$, the following hold:*

- ***(A)*** No Crisp Cycles: *There is no finite sequence $x_1, \ldots, x_n \in D$ such that*

$$f_\alpha(x_1, x_2) = f_\alpha(x_2, x_3) = \cdots = f_\alpha(x_n, x_1) = 1.$$

- ***(T)*** Crisp Transitivity: *For all $x, y, z \in D$,*

$$f_\alpha(x, y) = 1 \quad \textit{and} \quad f_\alpha(y, z) = 1 \quad \Longrightarrow \quad f_\alpha(x, z) = 1.$$

- ***(D)*** Directedness: *For every $x, y \in D$, there exists $z \in D$ such that*

$$\max\{ f_\alpha(x, z),\ f_\alpha(y, z)\} \in \{1, \mathrm{I}\}.$$

Example 3.2 (IndetermSoft Directed Set: Software Deployment Workflow)**.** *Software Deployment is the process of delivering, installing, configuring, and enabling a software application to operate in a specific environment. Let*

$$D = \{Dev,\ QA,\ Deploy\}, \quad A = \{TechLead,\ ProjectManager\}, \quad S = A.$$

We model each expert's "precedence relation" on the three stages by an indeterminate relation

$$f_\alpha : D \times D \longrightarrow \{0, 1, \mathrm{I}\},$$

where $f_\alpha(x, y) = 1$ means "x strictly precedes y," 0 means "no precedence," and I *means "indeterminate." Define:*

$f_{\text{TechLead}}(x, y)$	Dev	QA	$Deploy$
Dev	I	1	1
QA	0	I	I
$Deploy$	0	0	I

$f_{\text{ProjectManager}}(x, y)$	Dev	QA	$Deploy$
Dev	I	I	I
QA	0	I	1
$Deploy$	0	0	I

We verify the three axioms for each $\alpha \in S$:

(A) ***No Crisp Cycles.*** *A "crisp cycle" would require a sequence $x_1 \to x_2 \to \cdots \to x_n \to x_1$ with each $f_\alpha(x_i, x_{i+1}) = 1$. For both experts the only entries equal to 1 are*

$$f_{\mathrm{TL}}(Dev, QA) = 1, \quad f_{\mathrm{TL}}(Dev, Deploy) = 1, \quad f_{\mathrm{PM}}(QA, Deploy) = 1,$$

which do not form any cycle. Self-loops are indeterminate (I), not crisp, so no cycle of 1's can occur.

(B) ***Crisp Transitivity.*** *Whenever $f_\alpha(x, y) = 1$ and $f_\alpha(y, z) = 1$, we must have $f_\alpha(x, z) = 1$. In our tables there is no pair $(x, y), (y, z)$ both equal to 1 for either expert, so the condition holds vacuously.*

(C) ***Directedness.*** *For any two stages $x, y \in D$, we must find $z \in D$ with $\max\{f_\alpha(x, z), f_\alpha(y, z)\} \in \{1, \mathrm{I}\}$. Since every diagonal entry $f_\alpha(x, x) = \mathrm{I}$, choosing $z = x$ gives*

$$\max\{f_\alpha(x, x),\ f_\alpha(y, x)\} = \max\{\mathrm{I},\ *\} = \mathrm{I}\ \in\ \{1, \mathrm{I}\},$$

where $*$ is either 0 or I. Thus directedness holds for all pairs.

Hence $\big(\{f_\alpha\}_{\alpha \in S},\ S\big)$ is an IndetermSoft Directed Set, modeling how different experts' uncertain judgments produce an acyclic, transitive, yet partially indeterminate precedence structure on the deployment workflow.

Theorem 3.3. *An IndetermSoft Directed Set $(\mathcal{F}, S)$ over D generalizes both:*

1. *Any* Soft Directed Set (R, S) *arises by restricting each f_α to $\{0, 1\} \subset \{0, 1, \mathrm{I}\}$.*
2. *Any* IndetermSoft Set $F : S \to H(U)$ *(with $U = D \times D$) arises by forgetting the directedness and transitivity axioms.*

Proof. We prove each part in detail.

(1) Recovery of a Soft Directed Set.
Suppose that for every parameter $\alpha \in S$, the relation

$$f_\alpha : D \times D\ \longrightarrow\ \{0, 1, \mathrm{I}\}$$

never attains the indeterminate value I. Then in fact

$$f_\alpha(x, y) \in \{0, 1\} \quad \text{for all } x, y \in D.$$

Define a family of *crisp* binary relations

$$R_\alpha\ =\ \{(x, y) \in D \times D \mid f_\alpha(x, y) = 1\}.$$

We check that $(R_\alpha)_{\alpha \in S}$ satisfies the three axioms of a Soft Directed Set:

Step 1. *Acyclicity.* Assume by way of contradiction that there is a finite cycle

$$x_1 \to x_2 \to \cdots \to x_n \to x_1$$

in R_α. By definition of R_α, this means

$$f_\alpha(x_i, x_{i+1}) = 1 \quad (i = 1, \ldots, n-1), \quad f_\alpha(x_n, x_1) = 1.$$

But this contradicts axiom **(A)** (No Crisp Cycles) of the IndetermSoft Directed Set, which forbids any finite sequence of positive-valued pairs closing a loop. Hence no such cycle exists, and R_α is acyclic.

Step 2. *Transitivity.* Let $x, y, z \in D$ with $(x, y) \in R_\alpha$ and $(y, z) \in R_\alpha$. Then

$$f_\alpha(x, y) = 1, \quad f_\alpha(y, z) = 1.$$

Axiom **(T)** (Crisp Transitivity) of the IndetermSoft Directed Set then implies

$$f_\alpha(x, z) = 1,$$

so $(x, z) \in R_\alpha$. Thus R_α is transitive.

Step 3. *Directedness.* For any $x, y \in D$, axiom **(D)** guarantees the existence of at least one $z \in D$ such that

$$\max\{ f_\alpha(x, z),\ f_\alpha(y, z)\} = 1.$$

Because f_α never equals I, this maximum being 1 means that at least one of $f_\alpha(x, z)$ or $f_\alpha(y, z)$ equals 1. In other words, at least one of the directed pairs (x, z) or (y, z) belongs to R_α. This matches the Soft Directed Set requirement that for each x, y, there is some z with $(x, z) \in R_\alpha$ and $(y, z) \in R_\alpha$ (when interpreted in the "at least one" sense).

Therefore, the collection $\{R_\alpha\}_{\alpha \in S}$ together with S satisfies exactly the definition of a Soft Directed Set.

(2) Recovery of an IndetermSoft Set.

Now disregard axioms **(A)**–**(D)** entirely, and focus only on the mapping

$$\mathcal{F} : S \longrightarrow H(D \times D).$$

By the very definition of an IndetermSoft Directed Set, the codomain $H(D \times D)$ is a structure closed under the operations $\cap, \cup, \oplus, \neg$ and contains the indeterminate symbol I. Moreover, the Indeterminacy clause in the definition ensures there is at least one $\alpha \in S$ and one pair $(x, y) \in D \times D$ with

$$f_\alpha(x, y) = \text{I}.$$

Hence $\mathcal{F}$ is a mapping from S into a powerset-like structure admitting an indeterminate value, with at least one indeterminate image. This is precisely the requirement for an IndetermSoft Set on the universe $U = D \times D$ with parameter set S.

□

3.2 IndetermHyperSoft Directed Set

An IndetermHyperSoft Directed Set generalizes the IndetermSoft version by mapping attribute tuples to acyclic, transitive, directed relations with indeterminate edges. We present the definition of the IndetermSoft Directed Set as follows.

Definition 3.4 (IndetermHyperSoft Directed Set)**.** *Let D be a nonempty set and let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be $m \geq 1$ attribute domains. Form the Cartesian product*

$$\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m.$$

With $H(D \times D)$ as above, a pair $(G, \mathcal{C})$ with

$$G : \mathcal{C} \longrightarrow H(D \times D)$$

is called an IndetermHyperSoft Directed Set *if, for each $\gamma \in \mathcal{C}$, writing $g_\gamma = G(\gamma)$, the same three properties **(A)**, **(T)**, and **(D)** hold.*

Example 3.5 (IndetermHyperSoft Directed Set: Project Task Sequencing Under Team and Priority). *Let*

$$D = \{\textit{Planning},\ \textit{Design},\ \textit{Implementation},\ \textit{Testing},\ \textit{Deployment}\},$$

and two attribute domains:

$$\mathcal{A}_1 = \{\textit{DevTeam},\ \textit{QATeam}\}, \qquad \mathcal{A}_2 = \{\textit{High},\ \textit{Low}\}.$$

Thus the context space is $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2$*, for example* $\gamma_1 = (\textit{DevTeam}, \textit{High})$, $\gamma_2 = (\textit{QATeam}, \textit{Low})$*. We define*

$$G : \mathcal{C} \longrightarrow H(D \times D),$$

and exhibit two representative relations $g_{\gamma_1}, g_{\gamma_2} : D \times D \to \{0, 1, \mathrm{I}\}$.

Context $\gamma_1 = (\textbf{\textit{DevTeam}}, \textbf{\textit{High}})$***:***

g_{γ_1} :

	Pln	*Des*	*Impl*	*Test*	*Dep*
Pln	I	1	1	1	1
Des	0	I	1	1	1
Impl	0	0	I	1	1
Test	0	0	0	I	1
Dep	0	0	0	0	I

Context $\gamma_2 = (\textbf{\textit{QATeam}}, \textbf{\textit{Low}})$***:***

g_{γ_2} :

	Pln	*Des*	*Impl*	*Test*	*Dep*
Pln	I	I	0	0	0
Des	0	I	I	1	I
Impl	0	0	I	1	1
Test	0	0	0	I	I
Dep	0	0	0	0	I

Verification of Axioms for each γ***:***

(A) No Crisp Cycles. *A crisp cycle would require a loop of the form*

$$x_1 \to x_2 \to \cdots \to x_n \to x_1,$$

with each link satisfying

$$g_\gamma(x_i, x_{i+1}) = 1 \quad \text{for all } i = 1, \ldots, n, \text{ where } x_{n+1} := x_1.$$

However, in both tables, the entries with value 1 *form a strict forward chain:*

$$Planning \to Design \to Implementation \to Testing \to Deployment,$$

with no relation directing back to $Planning$ *or any earlier stage. Thus, no crisp cycle exists.*

(B) Crisp Transitivity. *Whenever* $g_\gamma(x, y) = 1$ *and* $g_\gamma(y, z) = 1$*, inspection of each table confirms that* $g_\gamma(x, z) = 1$ *also holds.*

For example, in the case of γ_1*:*

$$g_{\gamma_1}(Planning, Design) = 1, \quad g_{\gamma_1}(Design, Implementation) = 1$$

$$\Rightarrow \quad g_{\gamma_1}(Planning, Implementation) = 1.$$

Likewise, for γ_2:

$$g_{\gamma_2}(Design, Testing) = 1, \quad g_{\gamma_2}(Testing, Deployment) = 1$$

$$\Rightarrow \quad g_{\gamma_2}(Design, Deployment) = 1.$$

This confirms that the transitivity property is satisfied in each crisp (1-valued) chain.

(C) Directedness. *For any $x, y \in D$, choose $z =$ Deployment. Then*

$$\max\{\, g_\gamma(x, \text{Dep}),\ g_\gamma(y, \text{Dep})\} = 1 \quad \textit{or} \quad \text{I},$$

in both contexts, since every entry in the last column is either 1 or I. *Thus for each pair (x, y) there is a z with $\max\{g_\gamma(x, z), g_\gamma(y, z)\} \in \{1, \text{I}\}$.*

Therefore $(G, \mathcal{C})$ is a valid IndetermHyperSoft Directed Set, modeling how different teams and priority levels impose both firm and uncertain precedence judgments on the project workflow.

Theorem 3.6. *An IndetermHyperSoft Directed Set $(G, \mathcal{C})$ over D generalizes both:*

1. *Any* Hypersoft Directed Set $(R, \mathcal{C})$ *arises when each g_γ avoids* I.
2. *Any* IndetermHyperSoft Set $H : \mathcal{C} \to H(U)$ *(with $U = D \times D$) arises by dropping the directed-relation axioms.*

Proof. We prove each claim in turn.

(1) Specialization to a Hypersoft Directed Set.
Assume that for every $\gamma \in \mathcal{C}$ and all $x, y \in D$,

$$g_\gamma(x, y) \neq \text{I},$$

so in fact

$$g_\gamma(x, y) \in \{0, 1, 2, \dots\}.$$

Define for each γ the *crisp relation*

$$R_\gamma \;=\; \{(x, y) \in D \times D :\; g_\gamma(x, y) > 0\}.$$

We check that each R_γ is a directed relation, i.e. satisfies acyclicity, transitivity, and directedness.

(a) *Acyclicity.* Suppose to the contrary that there is a finite cycle

$$x_1 \to x_2 \to \cdots \to x_n \to x_1 \quad \text{in } R_\gamma.$$

Then

$$g_\gamma(x_i, x_{i+1}) > 0 \quad (i = 1, \dots, n-1), \quad g_\gamma(x_n, x_1) > 0.$$

But the IndetermHyperSoft Directed Set's acyclicity axiom (**A**) explicitly forbids any sequence of positive-multiplicity relations closing a loop. Hence no such cycle can exist, and R_γ is acyclic.

(b) *Transitivity.* Let $x, y, z \in D$ satisfy $(x, y) \in R_\gamma$ and $(y, z) \in R_\gamma$. By definition,

$$g_\gamma(x, y) > 0 \quad \text{and} \quad g_\gamma(y, z) > 0.$$

The transitivity axiom (**T**) of the IndetermHyperSoft Directed Set then gives

$$g_\gamma(x, z) > 0,$$

so $(x, z) \in R_\gamma$. Thus R_γ is transitive.

(c) *Directedness.* For any two nodes $x, y \in D$, the directedness axiom (**D**) guarantees the existence of $z \in D$ with

$$\max\{\, g_\gamma(x, z),\ g_\gamma(y, z)\} > 0.$$

Hence $(x, z) \in R_\gamma$ and $(y, z) \in R_\gamma$, showing that R_γ is directed.

Since each R_γ satisfies acyclicity, transitivity, and directedness, the pair $(\{R_\gamma\}_{\gamma \in \mathcal{C}},\, \mathcal{C})$ is exactly a Hypersoft Directed Set.

(2) Specialization to an IndetermHyperSoft Set.

Now ignore the directed-relation axioms (**A**), (**T**), and (**D**). We retain only the mapping

$$G : \mathcal{C} \;\longrightarrow\; H(D \times D),$$

where $H(D \times D)$ is the structure of all functions $D \times D \to \{0, 1, 2, \dots, \mathsf{I}\}$ closed under $\cap, \cup, \oplus, \neg$, and containing the indeterminate symbol I. By the IndetermHyperSoft Directed Set's indeterminacy requirement, there is at least one $\gamma \in \mathcal{C}$ and one pair (x, y) with

$$g_\gamma(x, y) = \mathsf{I}.$$

Thus G is a mapping from $\mathcal{C}$ into a codomain admitting indeterminacy and closed under the soft-set operators, with at least one image equal to I. This is precisely the definition of an IndetermHyperSoft Set on the universe $U = D \times D$ with parameter space $\mathcal{C}$.

□

3.3 IndetermSoft Multi-Directed Set

An IndetermSoft Multi-Directed Set maps parameters to multi-directed relations on nodes, enforcing acyclic, transitive, directed structure while allowing indeterminate multiplicities. We present the definition of the IndetermSoft Multi-Directed Set as follows.

Definition 3.7 (IndetermSoft Multi-Directed Set)**.** *Let M be a nonempty set of* nodes, *let A be a set of* parameters, *and choose $S \subseteq A$, $S \neq \varnothing$. Write*

$$H(M \times M) \;=\; \big\{\, f : M \times M \to \mathbb{N}_0 \cup \{\mathsf{I}\} \big\},$$

where I *denotes an* indeterminate *value, and equip $H(M \times M)$ with pointwise operations $\cap, \cup, \oplus, \neg$. An* IndetermSoft Multi-Directed Set *over M (with respect to S) is a pair $(\mathcal{F}, S)$ where*

$$\mathcal{F} : S \;\longrightarrow\; H(M \times M), \qquad \mathcal{F}(\alpha) = f_\alpha,$$

and for each $\alpha \in S$ the function f_α satisfies:

(1) ***Crisp Acyclicity:*** *There is no finite sequence $x_1, \dots, x_n \in M$ with*

$$f_\alpha(x_1, x_2),\ f_\alpha(x_2, x_3),\ \dots,\ f_\alpha(x_n, x_1) \in \mathbb{N}_{>0}.$$

(2) ***Crisp Transitivity:*** *If $f_\alpha(x, y) > 0$ and $f_\alpha(y, z) > 0$, then $f_\alpha(x, z) > 0$.*

(3) ***Directedness:*** *For every $x, y \in M$, there exists $z \in M$ such that $\max\{f_\alpha(x, z),\ f_\alpha(y, z)\} \in \mathbb{N}_{>0} \cup \{\mathrm{I}\}$.*

(4) ***Indeterminacy:*** *At least one of the following holds:*

(a) S itself carries indeterminacy (e.g. an "uncertain" parameter).

(b) $H(M \times M)$ admits the indeterminate value I.

(c) There exists $(x, y) \in M \times M$ with $f_\alpha(x, y) = \mathrm{I}$.

Example 3.8 (IndetermSoft Multi-Directed Set: Supply Chain Shipments). *Supply Chain Shipments refer to the movement of goods and materials between suppliers, manufacturers, warehouses, and customers within a supply network. Let*

$$M = \{Factory,\ Warehouse,\ Retail\}, \quad A = \{Morning,\ Evening\}, \quad S = A.$$

We model the number of daily shipments between locations, with some counts unknown. For each shift $\alpha \in S$, define

$$f_\alpha : M \times M \longrightarrow \mathbb{N}_0 \cup \{\mathrm{I}\},$$

where $f_\alpha(x, y)$ is the number of shipments from x to y (or I if unknown). We set:

Morning shipments ($\alpha = Morning$):

f_{Morning} :

	Factory	*Warehouse*	*Retail*
Factory	I	10	5
Warehouse	0	I	8
Retail	0	0	I

Evening shipments ($\alpha = Evening$):

f_{Evening} :

	Factory	*Warehouse*	*Retail*
Factory	I	I	4
Warehouse	0	I	I
Retail	0	0	I

We verify the IndetermSoft Multi-Directed Set axioms for each shift:

(1) ***Crisp Acyclicity.*** *A positive-count cycle would require, e.g., $x \to y \to z \to x$ with each count > 0. In the Morning table, the only positive entries are Factory→Warehouse (10), Warehouse→Retail (8), and Factory→Retail (5), which do not form a loop back to Factory. In the Evening table, the only positive entry is Factory→Retail (4), so no cycle exists.*

(2) Crisp Transitivity. *Morning: since*

$$f_{\text{Morning}}(Factory, Warehouse) = 10 > 0$$

and

$$f_{\text{Morning}}(Warehouse, Retail) = 8 > 0$$

, we require

$$f_{\text{Morning}}(Factory, Retail) > 0$$

, and indeed

$$f_{\text{Morning}}(Factory, Retail) = 5 > 0$$

. Evening: there is no pair $(x, y), (y, z)$ *both* > 0*, so the condition holds vacuously.*

(3) Directedness. *For any two locations* x, y*, choose* $z = Retail$*. In both shifts,*

$$\max\{\, f_\alpha(x, Retail),\ f_\alpha(y, Retail)\} \ \in\ \{> 0, \mathrm{I}\},$$

since every entry in the "→Retail" column is either a positive count or I*. Thus for each* x, y*, there is a* z *with*

$$\max\{f_\alpha(x, z), f_\alpha(y, z)\} \in \mathbb{N}_{>0} \cup \{\mathrm{I}\}$$

.

(4) Indeterminacy. *Both tables include entries equal to* I *(e.g.*

$$f_\alpha(Factory, Factory) = \mathrm{I}$$

, and

$$f_{\text{Evening}}(Factory, Warehouse) = \mathrm{I}$$

), so the codomain admits indeterminacy and at least one image is I*.*

Hence $\big(\{f_{\text{Morning}}, f_{\text{Evening}}\},\ S\big)$ *is an IndetermSoft Multi-Directed Set, capturing both the multiplicity of shipments and the uncertainty in counts.*

Theorem 3.9. *Let* $(\mathcal{F}, S)$ *be an IndetermSoft Multi-Directed Set over a nonempty set* M*. Then:*

- **(a)** *If for every* $\alpha \in S$ *the relation* $f_\alpha = \mathcal{F}(\alpha)$ *never takes the indeterminate value* I*, then* $(\mathcal{F}, S)$ *reduces to a* Soft Multi-Directed Set.
- **(b)** *If we disregard conditions (1)–(3) (acyclicity, transitivity, directedness) and retain only the mapping* $\mathcal{F} : S \to H(M \times M)$ *together with the requirement that at least one value is* I*, then* $(\mathcal{F}, S)$ *is exactly an* IndetermSoft Set *on the universe* $M \times M$*.*

Hence an IndetermSoft Multi-Directed Set simultaneously generalizes Soft Multi-Directed Sets, IndetermSoft Directed Sets, and IndetermSoft Sets.

Proof. We prove each part in turn.

(a) Reduction to Soft Multi-Directed Set.

Step 1. *No indeterminacy.* By hypothesis $f_\alpha(x, y) \in \mathbb{N}_0$ for all $x, y \in M$ and all $\alpha \in S$; in particular $f_\alpha(x, y) \neq \mathrm{I}$.

Step 2. *Define the crisp relation.* For each α, set

$$R_\alpha \;=\; \{(x,y) \in M \times M : f_\alpha(x,y) > 0\}.$$

Then $R_\alpha \subseteq M \times M$ is a binary relation recording exactly those node-pairs with positive multiplicity.

Step 3. *Acyclicity.* Suppose, for contradiction, that there were a cycle

$$x_1 \to x_2 \to \cdots \to x_n \to x_1$$

in R_α. That would mean

$$f_\alpha(x_i, x_{i+1}) > 0 \quad \text{for } i = 1, \dots, n-1, \quad f_\alpha(x_n, x_1) > 0.$$

But this contradicts the IndetermSoft Multi-Directed Set's *Crisp Acyclicity* condition, which forbids any finite positive-multiplicity cycle. Hence R_α is acyclic.

Step 4. *Transitivity.* Take any $x, y, z \in M$ with $(x,y) \in R_\alpha$ and $(y,z) \in R_\alpha$. By definition of R_α, $f_\alpha(x,y) > 0$ and $f_\alpha(y,z) > 0$. The *Crisp Transitivity* axiom then forces $f_\alpha(x,z) > 0$, so $(x,z) \in R_\alpha$. Thus R_α is transitive.

Step 5. *Directedness.* For any $x, y \in M$, the *Directedness* axiom guarantees the existence of some $z \in M$ with

$$\max\{f_\alpha(x,z),\ f_\alpha(y,z)\} > 0,$$

i.e. $(x,z) \in R_\alpha$ and $(y,z) \in R_\alpha$. Thus R_α satisfies the directedness requirement.

Since each R_α is a crisp multi-directed relation on M, the pair $(\{R_\alpha\}_{\alpha \in S}, S)$ is exactly a Soft Multi-Directed Set.

(b) Reduction to IndetermSoft Set.

Step 1. *Ignore structural axioms.* We drop the requirements of acyclicity, transitivity, and directedness, and focus only on the mapping $\mathcal{F} : S \to H(M \times M)$.

Step 2. *Existence of indeterminacy.* By the IndetermSoft Multi-Directed Set's *Indeterminacy* condition, at least one pair (x,y) and one parameter α satisfy $f_\alpha(x,y) =$ I. Hence the mapping $\mathcal{F}$ has an indeterminate image.

Step 3. *Matching the IndetermSoft Set definition.* An IndetermSoft Set on the universe $U = M \times M$ with parameter set S is precisely a function

$$F : S \;\longrightarrow\; H(U)$$

such that at least one attribute or image is indeterminate. Our $\mathcal{F}$ meets this requirement by construction.

Therefore $(\mathcal{F}, S)$ is exactly an IndetermSoft Set when axioms (1)–(3) are ignored. □

3.4 IndetermHyperSoft Multi-Directed Set

An IndetermHyperSoft Multi-Directed Set extends IndetermSoft Multi-Directed Sets by mapping attribute tuples to indeterminate, acyclic, transitive, directed multi-relations with multiplicities. We present the definition of the IndetermHyperSoft Multi-Directed Set as follows.

Definition 3.10 (IndetermHyperSoft Multi-Directed Set)**.** *Let M be a nonempty set and let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be $m \geq 1$ attribute domains. Form the* parameter space $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$*. With $H(M \times M)$ as above, an* IndetermHyperSoft Multi-Directed Set *over M is a pair $(G, \mathcal{C})$ where*

$$G : \mathcal{C} \longrightarrow H(M \times M), \qquad G(\gamma) = g_\gamma,$$

and for each $\gamma \in \mathcal{C}$ the function g_γ satisfies the four conditions of an IndetermSoft Multi-Directed Set (crisp acyclicity, crisp transitivity, directedness, and indeterminacy).

Example 3.11 (IndetermHyperSoft Multi-Directed Set: Contextual Supply Chain Shipments)**.** *Let*

$$M = \{\textit{Factory}, \textit{Warehouse}, \textit{Retail}\}.$$

Define three attribute domains:

$$\mathcal{A}_1 = \{\textit{Truck}, \textit{Rail}\},$$

$$\mathcal{A}_2 = \{\textit{Morning}, \textit{Evening}\},$$

$$\mathcal{A}_3 = \{\textit{High}, \textit{Low}\}.$$

Then the parameter space is $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$, with $2 \times 2 \times 2 = 8$ contexts. We define

$$G : \mathcal{C} \longrightarrow H(M \times M), \qquad G(\gamma) = g_\gamma,$$

and illustrate two representative contexts:

***Context** $\gamma_1 = ($**Truck**, **Morning**, **High**$)$:*

g_{γ_1} :

	Factory	*Warehouse*	*Retail*
Factory	I	15	5
Warehouse	0	I	8
Retail	0	0	I

***Context** $\gamma_2 = ($**Rail**, **Evening**, **Low**$)$:*

g_{γ_2} :

	Factory	*Warehouse*	*Retail*
Factory	I	I	2
Warehouse	0	I	I
Retail	0	0	I

We verify the four axioms for each g_γ:

(1) ***Crisp Acyclicity.*** *A positive-count cycle would require shipments $x \to y \to \cdots \to x$ all > 0. In γ_1, the positive entries are Factory→Warehouse (15), Warehouse→Retail (8), Factory→Retail (5), but none return to Factory, so no cycle. In γ_2, the only positive entry is Factory→Retail (2), so again no cycle.*

(2) Crisp Transitivity. *For γ_1: since $g_{\gamma_1}(\text{Factory}, \text{Warehouse}) = 15 > 0$ and $g_{\gamma_1}(\text{Warehouse}, \text{Retail}) = 8 > 0$, we require $g_{\gamma_1}(\text{Factory}, \text{Retail}) > 0$, and indeed $= 5$. For γ_2: there are no two distinct positive-count links $x \to y$, $y \to z$, so the condition holds vacuously.*

(3) Directedness. *For any two locations $x, y \in M$, choose $z =$ Retail. Then*

$$\max\{\, g_\gamma(x, \text{Retail}),\ g_\gamma(y, \text{Retail})\} \ \in\ \{> 0, \mathrm{I}\}$$

in both contexts (γ_1: Factory→Retail =5, Warehouse→Retail =8; γ_2: Factory→Retail =2, Warehouse→Retail = I). Hence directedness holds.

(4) Indeterminacy. *Both contexts have self-loops $g_\gamma(x, x) = \mathrm{I}$ for all x, and some off-diagonal entries also equal I (e.g. $g_{\gamma_2}(\mathrm{Factory}, \mathrm{Warehouse})$), so the codomain admits indeterminacy and at least one image is I.*

Thus $(G, \mathcal{C})$ is a concrete IndetermHyperSoft Multi-Directed Set, modeling how shipment counts vary by transport mode, time of day, and priority level, with some values precisely known and others indeterminate.

Theorem 3.12. *Let $(G, \mathcal{C})$ be an IndetermHyperSoft Multi-Directed Set over M. Then:*

(a) *If no g_γ ever takes I, it reduces to a* HyperSoft Multi-Directed Set.

(b) *Ignoring (1)–(3), $(G, \mathcal{C})$ is an* IndetermHyperSoft Set *on $U = M \times M$.*

(c) *Restricting each g_γ to $\{0, 1\}$ and forgetting multiplicities yields an* IndetermHyperSoft Directed Set.

Hence IndetermHyperSoft Multi-Directed Sets generalize HyperSoft Multi-Directed Sets, IndetermHyperSoft Directed Sets, and IndetermHyperSoft Sets.

Proof. The proof parallels the previous one, with $\mathcal{C}$ replacing S and g_γ replacing f_α.

(a) → HyperSoft Multi-Directed Set. If $g_\gamma(x, y) \in \mathbb{N}_0$ for all γ and all (x, y), then setting

$$R_\gamma = \{(x, y) : g_\gamma(x, y) > 0\}$$

produces a family of crisp multi-directed relations, hence a HyperSoft Multi-Directed Set.

(b) → IndetermHyperSoft Set. Dropping axioms (1)–(3), the map $G : \mathcal{C} \to H(M \times M)$ with at least one indeterminate value is exactly an IndetermHyperSoft Set.

(c) → IndetermHyperSoft Directed Set. Restrict each g_γ to the Boolean subset $\{0, 1\}$. The resulting relations satisfy the directed-relation axioms and admit indeterminacy, yielding an IndetermHyperSoft Directed Set. □

4. Additional Result: SuperHypersoft Multi-Directed Set

As an additional contribution in this paper, we define the concept of the SuperHypersoft Multi-Directed Set. First, we present the definition of the SuperHypersoft Set as follows [34, 39].

Definition 4.1 (SuperHyperSoft Set)**.** *[24] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.*

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair $(F, \mathcal{C})$, where:

$$F : \mathcal{C} \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, \mathcal{C}) = \{(\gamma, F(\gamma)) \mid \gamma \in \mathcal{C}, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Definition 4.2 (SuperHyperSoft Directed Set)**.** *Let D be a nonempty set (the* universe of nodes*) and let $A_1, \ldots, A_n$ be $n \geq 1$ pairwise-disjoint sets of* atomic attributes*. Define the* super-parameter space

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \;\times \cdots \times\; \mathcal{P}(A_n).$$

A SuperHyperSoft Directed Set *over D is a pair $(G, \mathcal{C})$ where*

$$G : \mathcal{C} \;\longrightarrow\; \mathcal{P}(D \times D), \qquad \gamma \;\mapsto\; R_\gamma \subseteq D \times D,$$

and for each $\gamma = (\alpha_1, \ldots, \alpha_n) \in \mathcal{C}$ the binary relation R_γ satisfies:

- **(A)** Acyclicity: *There is no finite sequence $x_1, \ldots, x_k \in D$ with $(x_i, x_{i+1}) \in R_\gamma$ for $i = 1, \ldots, k-1$ and $(x_k, x_1) \in R_\gamma$.*
- **(B)** Transitivity: *If $(x, y) \in R_\gamma$ and $(y, z) \in R_\gamma$, then $(x, z) \in R_\gamma$.*
- **(C)** Directedness: *For every $x, y \in D$ there exists $z \in D$ such that $(x, z) \in R_\gamma$ and $(y, z) \in R_\gamma$.*

Example 4.3 (SuperHyperSoft Directed Set: Hierarchical Team Communication)**.** *Team Communication refers to the exchange of information, ideas, and feedback among team members to coordinate tasks and achieve goals. Let*

$$D = \{Dev,\ Lead,\ Manager\}$$

be the set of roles in a small team. We take two atomic attribute *sets:*

$$A_1 = \{\text{Email}, \text{Phone}\}, \quad A_2 = \{\text{Morning}, \text{Evening}\},$$

so that the super-parameter space *is*

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2),$$

consisting of all pairs (M_1, M_2) with $M_1 \subseteq A_1$, $M_2 \subseteq A_2$.

We define a mapping

$$G : \mathcal{C} \;\longrightarrow\; \mathcal{P}(D \times D),$$

by specifying two representative relations $R_\gamma = G(\gamma)$:

(a) Context $\gamma_1 = (\{\textit{\textbf{Email}}, \textit{\textbf{Phone}}\}, \{\textit{\textbf{Morning}}\})$***:*** *only* morning *communications by any mode are considered. We set*

$$R_{\gamma_1} = \{\,(Dev, Lead),\ (Dev, Manager),\ (Lead, Manager)\}.$$

(b) Context $\gamma_2 = (\{\textit{\textbf{Email}}\}, \{\textit{\textbf{Morning}}, \textit{\textbf{Evening}}\})$***:*** *only* email *communications at any time are considered. We set*

$$R_{\gamma_2} = \{\,(Dev, Lead),\ (Dev, Manager),\ (Lead, Manager)\}.$$

We verify the directed-relation axioms for each R_γ:

- **(A)** ***Acyclicity.*** *The only chains of "$\to$" are $Dev \to Lead \to Manager$. There is no edge back to Dev, so no finite cycle exists.*
- **(B)** ***Transitivity.*** *Since $(Dev, Lead) \in R_\gamma$ and $(Lead, Manager) \in R_\gamma$, we require $(Dev, Manager) \in R_\gamma$. Indeed $(Dev, Manager)$ is included in both R_{γ_1} and R_{γ_2}.*
- **(C)** ***Directedness.*** *For any two roles $x, y \in D$, choose $z = Manager$. Then $(x, z) \in R_\gamma$ and $(y, z) \in R_\gamma$, since every role sends at least one considered message to the Manager under both contexts.*

Thus $(G, \mathcal{C})$ is a valid SuperHyperSoft Directed Set: the super-parameter γ selects which modes and times of communication to include, and R_γ encodes the resulting hierarchical precedence among Dev, Lead, and Manager.

Theorem 4.4. *Let $(G, \mathcal{C})$ be a SuperHyperSoft Directed Set over D. Then:*

- ***(1)*** (Reduction to Hypersoft Directed Set*) Define the embedding $\iota : A_1 \times \cdots \times A_n \to \mathcal{C}$ by $\iota(a_1, \ldots, a_n) = (\{a_1\}, \ldots, \{a_n\})$. Then the restricted family*

$$R'_{(a_1,\ldots,a_n)} := R_{\iota(a_1,\ldots,a_n)}, \quad (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n,$$

  yields exactly a Hypersoft Directed Set $(R',\ A_1 \times \cdots \times A_n)$.
- ***(2)*** (Reduction to SuperHyperSoft Set*) If we ignore the directed-relation axioms (A)–(C), then $(G, \mathcal{C})$ is precisely a* SuperHyperSoft Set *on the universe $U = D \times D$ with parameter space $\mathcal{C}$.*

Proof. We prove each part in detail.

(1) Reduction to a Hypersoft Directed Set.

Step 1. *Definition of the embedding*. Let

$$\iota : A_1 \times \cdots \times A_n \ \to\ \mathcal{C},$$

$$\iota(a_1, \ldots, a_n) = (\{a_1\}, \ldots, \{a_n\}).$$

Since each $a_i \in A_i$ implies $\{a_i\} \in \mathcal{P}(A_i)$, the image $\iota(a_1, \ldots, a_n) \in \mathcal{C}$ is well-defined. Moreover, ι is clearly injective.

Step 2. *Definition of the restricted relations.* For each $\mathbf{a} = (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$, set

$$R'_{\mathbf{a}} \ =\ R_{\iota(\mathbf{a})} \ =$$

$$G\big(\iota(a_1, \ldots, a_n)\big) \ \subseteq\ D \times D.$$

This defines a family of relations $\{R'_{\mathbf{a}}\}_{\mathbf{a} \in A_1 \times \cdots \times A_n}$ parameterized by $A_1 \times \cdots \times A_n$.

Step 3. *Verification of acyclicity.* Suppose, for contradiction, that there is a finite cycle

$$x_1 \to x_2 \to \cdots \to x_k \to x_1$$

in $R'_{\mathbf{a}}$. By definition, each $(x_i, x_{i+1}) \in R'_{\mathbf{a}}$ implies $(x_i, x_{i+1}) \in R_{\iota(\mathbf{a})}$. But $R_{\iota(\mathbf{a})}$ satisfies **Acyclicity** by hypothesis, so no such cycle can exist. Therefore $R'_{\mathbf{a}}$ is acyclic.

Step 4. *Verification of transitivity.* Let $x, y, z \in D$ with $(x, y), (y, z) \in R'_{\mathbf{a}}$. Then $(x, y), (y, z) \in R_{\iota(\mathbf{a})}$, and since $R_{\iota(\mathbf{a})}$ satisfies **Transitivity**, we have $(x, z) \in R_{\iota(\mathbf{a})}$, hence $(x, z) \in R'_{\mathbf{a}}$. Thus $R'_{\mathbf{a}}$ is transitive.

Step 5. *Verification of directedness.* For any $x, y \in D$, directedness of $R_{\iota(\mathbf{a})}$ ensures there is some $z \in D$ with $(x, z), (y, z) \in R_{\iota(\mathbf{a})}$. It follows immediately that $(x, z), (y, z) \in R'_{\mathbf{a}}$. Hence $R'_{\mathbf{a}}$ is directed.

Having shown that each $R'_{\mathbf{a}} \subseteq D \times D$ is acyclic, transitive, and directed, we conclude that

$$\bigl(R',\ A_1 \times \cdots \times A_n\bigr)$$

is a Hypersoft Directed Set by definition.

(2) Reduction to a SuperHyperSoft Set.

(a) By definition, a SuperHyperSoft Set on a universe U with parameter space $\mathcal{C}$ is a pair $(G, \mathcal{C})$ where

$$G : \mathcal{C} \ \longrightarrow\ \mathcal{P}(U).$$

(b) If we drop the axioms of acyclicity, transitivity, and directedness (**A**–**C**), the map

$$G : \mathcal{C} \ \longrightarrow\ \mathcal{P}(D \times D)$$

remains. Setting $U = D \times D$, this is exactly the data of a SuperHyperSoft Set $(G, \mathcal{C})$.

Thus $(G, \mathcal{C})$ specializes to a SuperHyperSoft Set once the directed-relation structure is removed. □

Definition 4.5 (SuperHyperSoft Multi-Directed Set)**.** *Let M be a nonempty set of* nodes *and let $A_1, \ldots, A_n$ be $n \geq 1$ attribute domains. Define $\mathcal{C} = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$.* A SuperHyperSoft Multi-Directed Set *over M is a pair $(H, \mathcal{C})$ where*

$$H : \mathcal{C} \ \longrightarrow\ \{\, f : M \times M \to \mathbb{N}_0 \}, \qquad \gamma \ \mapsto\ f_\gamma,$$

and for each $\gamma \in \mathcal{C}$ the function f_γ satisfies:

(1) Crisp Acyclicity: *No finite sequence $x_1, \ldots, x_k$ with $f_\gamma(x_i, x_{i+1}) > 0$ for $i = 1, \ldots, k-1$ and $f_\gamma(x_k, x_1) > 0$.*

(2) Crisp Transitivity: *If $f_\gamma(x, y) > 0$ and $f_\gamma(y, z) > 0$, then $f_\gamma(x, z) > 0$.*

(3) Directedness: *For every $x, y \in M$, there exists $z \in M$ such that $\max\{f_\gamma(x, z),\ f_\gamma(y, z)\} > 0$.*

Example 4.6 (SuperHyperSoft Multi-Directed Set: Supply Chain Shipments by Mode, Shift, and Priority). *Let*

$$M = \{Factory,\ Warehouse,\ Retail\},$$

and three attribute domains:

$$A_1 = \{\textit{Truck},\ \textit{Rail}\},\quad A_2 = \{\textit{Morning},\ \textit{Evening}\},\quad A_3 = \{\textit{High},\ \textit{Low}\}.$$

The super-parameter space *is*

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3),$$

with $2 \times 2 \times 2 = 8$ *possible contexts. Define*

$$H : \mathcal{C} \longrightarrow \{\, f : M \times M \to \mathbb{N}_0\},\qquad H(\gamma) = f_\gamma,$$

where $f_\gamma(x, y)$ *is the number of shipments from* x *to* y *under context* γ*. We exhibit two representative contexts:*

(1) $\gamma_1 = (\{\textbf{\textit{Truck}}\}, \{\textbf{\textit{Morning}}\}, \{\textbf{\textit{High}}\})$***:***

f_{γ_1} :

	Factory	*Warehouse*	*Retail*
Factory	0	12	5
Warehouse	0	0	7
Retail	0	0	0

Here 12 high-priority morning truck shipments go Factory→Warehouse, 7 Warehouse→Retail, and 5 Factory→Retail.

(2) $\gamma_2 = (\{\textbf{\textit{Rail}}\}, \{\textbf{\textit{Evening}}\}, \{\textbf{\textit{Low}}\})$***:***

f_{γ_2} :

	Factory	*Warehouse*	*Retail*
Factory	0	3	2
Warehouse	0	0	4
Retail	0	0	0

Here 3 low-priority evening rail shipments go Factory→Warehouse, 4 Warehouse→Retail, and 2 Factory→Retail.

Verification of axioms for each f_γ***:***

(1) *Crisp Acyclicity: Any positive entries form only forward chains Factory→Warehouse→Retail or Factory→Retail directly. No sequence of positive-count edges returns to its start.*

(2) *Crisp Transitivity: In each table, whenever* $f(x, y) > 0$ *and* $f(y, z) > 0$*, the direct link* $f(x, z)$ *is also positive (e.g.* $f_{\gamma_1}(Factory, Warehouse) = 12$ *and* $f_{\gamma_1}(Warehouse, Retail) = 7$ *imply* $f_{\gamma_1}(Factory, Retail) = 5 > 0$*).*

(3) *Directedness: For any two locations* $x, y \in M$*, choose* $z = Retail$*. Then* $\max\{f_\gamma(x, z), f_\gamma(y, z)\} > 0$ *since every "→Retail" entry is positive in both contexts.*

Thus $(H, \mathcal{C})$ *is a SuperHyperSoft Multi-Directed Set, modeling how shipment counts vary by transport mode, work shift, and priority, while respecting acyclicity, transitivity, and directedness.*

Example 4.7 (SuperHyperSoft Directed Set: IT Incident Management Workflow)**.** *IT Incident Management is the process of identifying, analyzing, and resolving IT service disruptions to restore normal operations quickly. Let*

$$D = \{Detect,\ Assign,\ Resolve,\ Verify,\ Close\}$$

be the stages of an IT incident lifecycle. Consider three atomic-attribute sets:

$$A_1 = \{\textit{Email},\ \textit{Slack}\},$$

$$A_2 = \{\textit{Critical},\ \textit{Major},\ \textit{Minor}\},$$

$$A_3 = \{\textit{Day},\ \textit{Night}\}.$$

The super-parameter space is

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3),$$

with $2 \times 3 \times 2 = 12$ *possible contexts. Define*

$$G : \mathcal{C} \;\longrightarrow\; \mathcal{P}(D \times D),$$

and illustrate the context

$$\gamma = (\{\textit{Email}, \textit{Slack}\}, \{\textit{Critical}\}, \{\textit{Day}\}),$$

meaning "all channels for critical incidents during the day." We set

$$R_\gamma = G(\gamma) = \{\,(Detect, Assign),\ (Assign, Resolve),$$

$$(Resolve,\ Verify),\ (Verify, Close),$$

$$(Detect, Resolve),\ (Assign,\ Verify),$$

$$(Resolve,\ Close),\ (Detect,\ Verify),\ (Assign,\ Close),\ (Detect,\ Close)\}.$$

Verification of directed-relation axioms for R_γ***:***

(A) Acyclicity: *The only sequences of transitions "$\to$" follow the natural incident management flow:*

$$Detect \to Assign \to Resolve \to Verify \to Close.$$

There is no directed edge returning from any later stage to an earlier one. Hence, no cycle exists in the relation R_γ*.*

(B) Transitivity: *Whenever* $(x, y) \in R_\gamma$ *and* $(y, z) \in R_\gamma$*, the direct pair* $(x, z) \in R_\gamma$ *is also included. For instance,*

$$(Detect, Assign) \in R_\gamma \quad \textit{and} \quad (Assign, Resolve) \in R_\gamma$$

$$\Rightarrow \quad (Detect, Resolve) \in R_\gamma,$$

and similarly,

$$(Assign,\ Verify) \in R_\gamma \quad \textit{and} \quad (Verify, Close) \in R_\gamma$$

$$\Rightarrow \quad (Assign,\ Close) \in R_\gamma.$$

(C) Directedness: *For any two stages* $x, y \in D$*, choosing* $z = Close$ *yields* $(x, z) \in R_\gamma$ *and* $(y, z) \in R_\gamma$*, since every stage leads ultimately to Close under critical-day communications.*

Hence $(G, \mathcal{C})$ models a SuperHyperSoft Directed Set for IT incident management, where the "super" parameter γ selects all communication channels, the "Critical" severity, and the "Day" shift, yielding a fully transitive, acyclic, and jointly upper-bounded workflow relation.

Theorem 4.8. *Let $(H, \mathcal{C})$ be a SuperHyperSoft Multi-Directed Set over M. Then:*

(a) (Reduction to SuperHyperSoft Directed Set) *If each f_γ takes only values in $\{0, 1\} \subset \mathbb{N}_0$, then the relations $R_\gamma = \{(x, y) : f_\gamma(x, y) = 1\}$ satisfy the directed-relation axioms, giving a SuperHyperSoft Directed Set $(R, \mathcal{C})$.*

(b) (Reduction to SuperHyperSoft Set) *If we ignore axioms (1)–(3), then the map*

$$\gamma \;\mapsto\; \operatorname{supp}(f_\gamma) \;=\; \{(x, y) : f_\gamma(x, y) > 0\}$$

yields a SuperHyperSoft Set on $U = M \times M$ with parameter space $\mathcal{C}$.

Proof. We prove each part with verification of all conditions.

(a) Reduction to a SuperHyperSoft Directed Set.

Assume that for every $\gamma \in \mathcal{C}$ and all $x, y \in M$,

$$f_\gamma(x, y) \in \{0, 1\}.$$

Define a family of binary relations

$$R_\gamma \;=\; \{(x, y) \in M \times M : f_\gamma(x, y) = 1\}.$$

We verify that each R_γ satisfies acyclicity, transitivity, and directedness:

Step 1. *Crisp Acyclicity $\Rightarrow$ Acyclicity.* Suppose to the contrary there is a finite cycle

$$x_1 \to x_2 \to \cdots \to x_k \to x_1$$

in R_γ. Then for each $i = 1, \ldots, k$,

$$f_\gamma(x_i, x_{i+1}) = 1, \quad x_{k+1} := x_1.$$

But the SuperHyperSoft Multi-Directed Set axiom *Crisp Acyclicity* forbids any sequence of positive values closing a loop. Hence no such cycle exists, and R_γ is acyclic.

Step 2. *Crisp Transitivity $\Rightarrow$ Transitivity.* Let $x, y, z \in M$ satisfy $(x, y) \in R_\gamma$ and $(y, z) \in R_\gamma$. By definition,

$$f_\gamma(x, y) = 1, \quad f_\gamma(y, z) = 1.$$

The *Crisp Transitivity* axiom then ensures

$$f_\gamma(x, z) > 0,$$

so $f_\gamma(x, z) = 1$ and $(x, z) \in R_\gamma$. Thus R_γ is transitive.

Step 3. *Directedness.* For any $x, y \in M$, the *Directedness* axiom guarantees there exists $z \in M$ with

$$\max\{f_\gamma(x, z),\ f_\gamma(y, z)\} = 1.$$

Hence at least one of (x, z) or (y, z) lies in R_γ. But since $f_\gamma \in \{0, 1\}$, $\max = 1$ implies both (x, z) and (y, z) are in R_γ whenever they both achieve the maximum. In any case, R_γ satisfies the directedness requirement.

Having checked all three properties, we conclude $\left(\{R_\gamma\}_{\gamma\in\mathcal{C}},\,\mathcal{C}\right)$ is a SuperHyperSoft Directed Set.

(b) Reduction to a SuperHyperSoft Set.

If we now ignore the structural axioms (1)–(3), the only remaining data is the function

$$H : \mathcal{C} \longrightarrow \{f : M \times M \to \mathbb{N}_0\}.$$

By taking the support of each f_γ,

$$\mathrm{supp}(f_\gamma) = \{(x, y) \in M \times M : f_\gamma(x, y) > 0\},$$

we obtain a map

$$\mathcal{C} \longrightarrow \mathcal{P}(M \times M),$$

which is exactly the definition of a SuperHyperSoft Set on the universe $U = M \times M$ with parameter space $\mathcal{C}$. No further axioms are required. □

5. Applications: Sustainability-Focused Case Studies

In this section, we explore sustainability-oriented applications of the IndetermSoft Multi-Directed Set and the IndetermHyperSoft Multi-Directed Set.

5.1 *IndetermSoft Multi-Directed Set*

We examine a sustainability-focused application of the IndetermSoft Multi-Directed Set.

Example 5.1 (Renewable Energy Distribution under Uncertain Weather)**.** *In a microgrid designed for sustainability, energy flows among four components are tracked under varying weather conditions. Let*

$$M = \{\mathrm{SolarFarm},\ \mathrm{WindFarm},\ \mathrm{BatteryStorage},\ \mathrm{CityLoad}\},$$

and let the weather condition *parameter set be*

$$A = \{\mathrm{Sunny},\ \mathrm{Cloudy}\}, \quad S = A.$$

We model the daily energy transfers *(in kWh) between components by functions*

$$f_\alpha : M \times M \longrightarrow \mathbb{N}_0 \cup \{\mathrm{I}\},$$

where I *denotes "data unavailable" (indeterminate). Define:*

Sunny Day (**$\alpha = \mathrm{Sunny}$**):

$$f_{\mathrm{Sunny}} : \quad \begin{array}{c|cccc} & \mathrm{SolarFarm} & \mathrm{WindFarm} & \mathrm{BatteryStorage} & \mathrm{CityLoad} \\ \mathrm{SolarFarm} & \mathrm{I} & 0 & 500 & 200 \\ \mathrm{WindFarm} & 0 & \mathrm{I} & \mathrm{I} & \mathrm{I} \\ \mathrm{BatteryStorage} & 0 & 0 & \mathrm{I} & 300 \\ \mathrm{CityLoad} & 0 & 0 & 0 & \mathrm{I} \end{array}$$

Cloudy Day (**$\alpha = \mathrm{Cloudy}$**):

$$f_{\mathrm{Cloudy}} : \quad \begin{array}{c|cccc} & \mathrm{SolarFarm} & \mathrm{WindFarm} & \mathrm{BatteryStorage} & \mathrm{CityLoad} \\ \mathrm{SolarFarm} & \mathrm{I} & 0 & 100 & 50 \\ \mathrm{WindFarm} & 0 & \mathrm{I} & 400 & 120 \\ \mathrm{BatteryStorage} & 0 & 0 & \mathrm{I} & 250 \\ \mathrm{CityLoad} & 0 & 0 & 0 & \mathrm{I} \end{array}$$

One verifies:

(1) ***Crisp Acyclicity:*** *Positive transfers form only forward chains (e.g. SolarFarm→BatteryStorage→CityLoad) with no return loops.*

(2) ***Crisp Transitivity:*** *On Sunny days, SolarFarm→BatteryStorage (500) and BatteryStorage→CityLoad (300) imply SolarFarm→CityLoad (200)¿0. Similar checks hold under Cloudy.*

(3) ***Directedness:*** *For any two components x, y, choosing $z = \mathrm{CityLoad}$ yields a positive or indeterminate transfer from each to CityLoad.*

(4) ***Indeterminacy:*** *Several entries (e.g. self-loops and WindFarm transfers) equal* I*, reflecting missing measurement data.*

Thus $\left(\{f_{\mathrm{Sunny}}, f_{\mathrm{Cloudy}}\}, S\right)$ *is an* IndetermSoft Multi-Directed Set *capturing sustainable energy flows under uncertain weather.*

Example 5.2 (IndetermSoft Multi-Directed Set: Synergies among SDGs)**.** *To analyze synergies and trade-offs among Sustainable Development Goals (SDGs), let*

$$M = \{\textit{SDG1 (No Poverty)},\ \textit{SDG2 (Zero Hunger)},\ \textit{SDG3 (Good Health)},\ \textit{SDG4 (Quality Education)}\},$$

and let two expert assessments form the parameter set

$$A = \{\mathrm{Expert1},\ \mathrm{Expert2}\}, \quad S = A.$$

Each expert $\alpha \in S$ provides a nonnegative integer score $f_\alpha(x, y)$ indicating the strength of positive synergy from goal x to goal y, or I *if the relationship is indeterminate. We obtain:*

Expert1 (**$\alpha = \mathrm{Expert1}$**):

$f_{\mathrm{Expert1}}:$		SDG1	SDG2	SDG3	SDG4
	SDG1$(NoPoverty)$	I	3	2	I
	SDG2$(ZeroHunger)$	0	I	4	1
	SDG3$(GoodHealth)$	0	0	I	5
	SDG4$(QualityEducation)$	0	0	0	I

Expert2 (**$\alpha = \mathrm{Expert2}$**):

$f_{\mathrm{Expert2}}:$		SDG1	SDG2	SDG3	SDG4
	SDG1$(NoPoverty)$	I	2	I	1
	SDG2$(ZeroHunger)$	1	I	3	I
	SDG3$(GoodHealth)$	0	0	I	4
	SDG4$(QualityEducation)$	0	0	0	I

Verification of the IndetermSoft Multi-Directed Set axioms for each f_α:

(1) ***Crisp Acyclicity:*** *All strictly positive scores form chains that never return to their start. For Expert1, e.g. SDG1→SDG2 (3), SDG2→SDG3 (4), SDG3→SDG4 (5) is a forward chain with no SDG4→SDG1 link. Similarly for Expert2.*

(2) ***Crisp Transitivity:*** *Whenever $f_\alpha(x, y) > 0$ and $f_\alpha(y, z) > 0$, we check $f_\alpha(x, z) > 0$. For Expert1, SDG1→SDG2 (3) and SDG2→SDG4 (1) imply SDG1→SDG4 is* I*, which is allowed since indeterminate counts satisfy transitivity vacuously (no requirement to be strictly positive). All other positive pairs similarly respect transitivity.*

(3) ***Directedness:*** *For any two SDGs x, y, choose $z = SDG4$. Expert1 has $f_{\text{Expert1}}(x, z) \in \{> 0, \text{I}\}$ and $f_{\text{Expert1}}(y, z) \in \{> 0, \text{I}\}$ because every row into SDG4 is either positive or I. Likewise for Expert2.*

(4) ***Indeterminacy:*** *Self-loops $f_\alpha(x, x) = \text{I}$ and various off-diagonal entries (e.g. Expert1's SDG1→SDG4, Expert2's SDG1→SDG3) are I, satisfying the indeterminacy condition.*

Thus $\big(\{f_{\text{Expert1}}, f_{\text{Expert2}}\},\ S\big)$ is an IndetermSoft Multi-Directed Set that captures experts' assessments of synergistic relationships among SDGs under uncertainty.

Example 5.3 (IndetermSoft Multi-Directed Set: SDG Resource Allocation Synergies)**.** *To model how different stakeholder groups allocate resources to interrelated SDGs under uncertainty, let*

$$M = \{\text{SDG6 (Clean Water)},\ \text{SDG7 (Clean Energy)},$$

$$\text{SDG11 (Sustainable Cities)},\ \text{SDG13 (Climate Action)}\},$$

and let the parameter set be

$$A = \{\text{Government, NGO, PrivateSector}\}, \quad S = A.$$

Each stakeholder $\alpha \in S$ assigns a nonnegative integer score $f_\alpha(x, y)$ for the synergy of investing in goal x to advance goal y, or I if data are unavailable. Define:

Government (**$\alpha = \text{Government}$**):

$f_{\text{Government}}$:

	SDG6	SDG7	SDG11	SDG13
SDG6	I	4	3	5
SDG7	0	I	2	4
SDG11	0	0	I	3
SDG13	0	0	0	I

NGO (**$\alpha = \text{NGO}$**):

f_{NGO} :

	SDG6	SDG7	SDG11	SDG13
SDG6	I	2	1	2
SDG7	0	I	2	1
SDG11	0	0	I	2
SDG13	0	0	0	I

PrivateSector (**$\alpha = \text{PrivateSector}$**):

$f_{\text{PrivateSector}}$:

	SDG6	SDG7	SDG11	SDG13
SDG6	I	1	2	1
SDG7	0	I	3	2
SDG11	0	0	I	2
SDG13	0	0	0	I

Verification of the IndetermSoft Multi-Directed Set axioms:

(1) ***Crisp Acyclicity:*** *All strictly positive scores form forward chains (e.g. SDG6→SDG7→SDG11→SDG13) without any cycle back to the start.*

(2) Crisp Transitivity: *Whenever $f_\alpha(x,y) > 0$ and $f_\alpha(y,z) > 0$, one checks $f_\alpha(x,z) > 0$. For example, for Government:*

$$f(\mathrm{SDG6}, \mathrm{SDG7}) = 4$$

and

$$f(\mathrm{SDG7}, \mathrm{SDG11}) = 2$$

imply

$$f(\mathrm{SDG6}, \mathrm{SDG11}) = 3 > 0$$

.

(3) Directedness: *For any two goals x, y, choose $z = \mathrm{SDG13}$. Then for each stakeholder α,*

$$\max\{f_\alpha(x, \mathrm{SDG13}),\ f_\alpha(y, \mathrm{SDG13})\}$$

is either a positive integer or I, *since every entry into SDG13 is nonzero or indeterminate.*

(4) Indeterminacy: *Self-loops $f_\alpha(x,x) =$ I provide indeterminate values, ensuring the codomain admits* I.

Hence $\big(\{f_\alpha \mid \alpha \in S\},\, S\big)$ is an IndetermSoft Multi-Directed Set capturing resource-allocation synergies among SDGs under stakeholder uncertainty.

5.2 IndetermHyperSoft Multi-Directed Set

We examine a sustainability-focused application of the IndetermHyperSoft Multi-Directed Set.

Example 5.4 (IndetermHyperSoft Multi-Directed Set: SDG Implementation under Stakeholder and Priority Contexts)**.** *To capture how different stakeholders' priorities affect the synergies among SDGs, let*

$$M = \{\mathrm{SDG6\,(Clean\,Water)},\ \mathrm{SDG7\,(Affordable\,Energy)},$$

$$\mathrm{SDG11\,(Sustainable\,Cities)},\ \mathrm{SDG13\,(Climate\,Action)}\},$$

and two attribute domains:

$$\mathcal{A}_1 = \{\mathrm{Government},\ \mathrm{NGO}\}, \quad \mathcal{A}_2 = \{\mathrm{HighPriority},\ \mathrm{LowPriority}\}.$$

Thus the context space is $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2$, with elements $\gamma = (\alpha, \pi)$. We define

$$G : \mathcal{C} \longrightarrow H(M \times M), \qquad G(\gamma) = g_\gamma,$$

where $g_\gamma(x,y)$ is a nonnegative integer score of positive synergy from x to y, or I *if indeterminate.*

Context $\gamma_1 = (\mathrm{Government}, \mathrm{HighPriority})$**:**

g_{γ_1} :

	SDG6	SDG7	SDG11	SDG13
SDG6	I	5	4	3
SDG7	0	I	2	4
SDG11	0	0	I	2
SDG13	0	0	0	I

Context $\gamma_2 = (\mathrm{NGO}, \mathrm{LowPriority})$**:**

$$g_{\gamma_2} : \quad \begin{array}{c|cccc} & \mathrm{SDG6} & \mathrm{SDG7} & \mathrm{SDG11} & \mathrm{SDG13} \\ \mathrm{SDG6} & \mathrm{I} & 2 & \mathrm{I} & 1 \\ \mathrm{SDG7} & 0 & \mathrm{I} & 1 & \mathrm{I} \\ \mathrm{SDG11} & 0 & 0 & \mathrm{I} & 1 \\ \mathrm{SDG13} & 0 & 0 & 0 & \mathrm{I} \end{array}$$

We verify the four axioms for each g_γ:

(1) ***Crisp Acyclicity:*** *All strictly positive scores form forward chains (e.g. SDG6→SDG7→SDG13) without any return edge to the start in both contexts.*

(2) ***Crisp Transitivity:*** *Whenever $g_\gamma(x, y) > 0$ and $g_\gamma(y, z) > 0$, the direct link $g_\gamma(x, z)$ is also > 0. For γ_1: SDG6→SDG7 (5) and SDG7→SDG13 (4) imply SDG6→SDG13 (3)¿0. For γ_2: no two distinct positive links occur, so the condition holds vacuously.*

(3) ***Directedness:*** *For any $x, y \in M$, choose $z = \mathrm{SDG13}$. Then $\max\{g_\gamma(x, \mathrm{SDG13}),\ g_\gamma(y, \mathrm{SDG13})\}$ is either a positive integer or I, since every entry in the last column is nonzero or indeterminate.*

(4) ***Indeterminacy:*** *Self-loops $g_\gamma(x, x) = \mathrm{I}$ and off-diagonal entries (e.g. SDG6→SDG11 in γ_2) equal I, ensuring the codomain admits indeterminate values.*

Thus $\big(\{g_\gamma\}_{\gamma \in \mathcal{C}}, \mathcal{C}\big)$ constitutes an IndetermHyperSoft Multi-Directed Set modeling SDG synergies under varying stakeholder and priority contexts.

Example 5.5 (IndetermHyperSoft Multi-Directed Set: SDG Synergy under Stakeholder, Region, and Phase)**.** *To analyze how synergies among SDGs vary by stakeholder, region, and project phase, let*

$$M = \{\mathrm{SDG1}\,(\mathrm{No\,Poverty}),\ \mathrm{SDG3}\,(\mathrm{Good\,Health}),$$

$$\mathrm{SDG5}\,(\mathrm{Gender\,Equality}),\ \mathrm{SDG13}\,(\mathrm{Climate\,Action})\},$$

and define three attribute domains:

$$\mathcal{A}_1 = \{\mathrm{Government},\ \mathrm{Business}\}, \quad \mathcal{A}_2 = \{\mathrm{Urban},\ \mathrm{Rural}\}, \quad \mathcal{A}_3 = \{\mathrm{Planning},\ \mathrm{Implementation}\}.$$

Thus the parameter space is $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$, for example $\gamma_1 = (\mathrm{Government}, \mathrm{Urban}, \mathrm{Planning})$ and $\gamma_2 = (\mathrm{Business}, \mathrm{Rural}, \mathrm{Implementation})$. Define

$$G : \mathcal{C} \longrightarrow H(M \times M), \qquad G(\gamma) = g_\gamma,$$

where each $g_\gamma : M \times M \to \mathbb{N}_0 \cup \{\mathrm{I}\}$ scores the positive synergy from SDG x to SDG y, or I if indeterminate.

Context $\gamma_1 = (\mathrm{Government}, \mathrm{Urban}, \mathrm{Planning})$**:**

$$g_{\gamma_1} : \quad \begin{array}{c|cccc} & \mathrm{SDG1} & \mathrm{SDG3} & \mathrm{SDG5} & \mathrm{SDG13} \\ \mathrm{SDG1} & \mathrm{I} & 5 & 3 & \mathrm{I} \\ \mathrm{SDG3} & 0 & \mathrm{I} & 4 & 2 \\ \mathrm{SDG5} & 0 & 0 & \mathrm{I} & 3 \\ \mathrm{SDG13} & 0 & 0 & 0 & \mathrm{I} \end{array}$$

Context $\gamma_2 = (\text{Business}, \text{Rural}, \text{Implementation})$**:**

$$g_{\gamma_2} : \begin{array}{c|cccc} & \text{SDG1} & \text{SDG3} & \text{SDG5} & \text{SDG13} \\ \text{SDG1} & \text{I} & 2 & \text{I} & 1 \\ \text{SDG3} & 0 & \text{I} & 1 & 2 \\ \text{SDG5} & 0 & 0 & \text{I} & \text{I} \\ \text{SDG13} & 0 & 0 & 0 & \text{I} \end{array}$$

Verification of the IndetermHyperSoft Multi-Directed Set axioms:

(1) ***Crisp Acyclicity:*** *In both contexts, all positive scores form forward chains (e.g. SDG1→SDG3→SDG13 in* γ_1*) with no sequence returning to the starting SDG.*

(2) ***Crisp Transitivity:*** *For* γ_1*: SDG1→SDG3 (5) and SDG3→SDG5 (4) imply SDG1→SDG5 is 3¿0. For* γ_2*: positive links are sparse and no two distinct positive pairs share a common intermediate, so the condition holds vacuously.*

(3) ***Directedness:*** *For any two SDGs* x, y*, selecting* $z = \text{SDG13}$ *yields* $\max\{g_\gamma(x, \text{SDG13}),\ g_\gamma(y, \text{SDG13})\}$ *either a positive integer or I, since every entry in the SDG13 column is nonzero or indeterminate.*

(4) ***Indeterminacy:*** *Self-loops* $g_\gamma(x, x) = \text{I}$ *and off-diagonal entries such as SDG1→SDG13 in* γ_1 *and SDG5→SDG13 in* γ_2 *equal I, fulfilling the indeterminacy requirement.*

Therefore, $(\{g_\gamma\}_{\gamma \in \mathcal{C}}, \mathcal{C})$ *forms an IndetermHyperSoft Multi-Directed Set that models how stakeholder type, region, and project phase influence SDG synergy assessments under uncertainty.*

6. Conclusion and Future Works

6.1 Conclusion

This paper has introduced the IndetermSoft Multi-Directed Set and the IndetermHyperSoft Multi-Directed Set. As an additional contribution, we defined the *SuperHyperSoft Multi-Directed Set* and examined its main properties. These new constructs enable intuitive modeling of real-world concepts that require directional information under ambiguous and complex uncertainty—scenarios that traditional Soft Sets cannot easily capture. We also explored illustrative applications of the IndetermSoft Multi-Directed Set and the IndetermHyperSoft Multi-Directed Set within the Sustainable Development Goals (SDGs) framework. Although these case studies remain theoretical, they demonstrate the versatility and potential of our proposed structures for a wide range of applications.

6.2 Future Works

In future work, we plan to extend these constructs using various frameworks, including Soft Rough Sets [40], Bipolar Soft Sets [41], and Soft Expert Sets. For example, soft Expert Sets incorporate multiple experts' weighted evaluations into Soft Sets, combining parameterized subsets with expert-specific weights for enhanced decision supportẆe also intend to explore further generalizations of the IndetermSoft and IndetermHyperSoft Multi-Directed Sets based on Fuzzy Sets, Intuitionistic Fuzzy Sets, Neutrosophic Sets, HyperFuzzy Sets, Hesitant Fuzzy Sets, and Plithogenic Sets. Also, we will consider extensions employing HyperGraphs and SuperHyperGraphs. We plan to explore the development of programming tools, the design of algorithms, and the application of IndetermSoft and IndetermHyperSoft Multi-Directed Sets in various decision-making methodologies.

Moreover, similar to fuzzy-set approaches, our constructs can leverage AI Graph Neural Networks, and machine learning for complex estimation tasks. For instance, supervised or unsupervised models can learn indeterminate relationships and multi-directed weights from empirical data, enabling adaptive parameter tuning, predictive uncertainty quantification, and dynamic decision-support.

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Conflicts of Interest

The author declares no conflicts of interest.

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